Investigations in Planar Physics

by

Yusuf İpekoğlu

B.Sc., Middle East Technical University, 1985
M.Sc., Middle East Technical University, 1988

Submitted to the Department of Physics
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology

September 1994

Signature of Author ______________________________
Department of Physics
June 22, 1994

Certified by ______________________________
Professor Roman Jackiw
Thesis Supervisor

Accepted by ______________________________
Professor George F. Koster
Chairman, Physics Graduate Committee

Investigations in Planar Physics

by

Yusuf İpekoğlu

Submitted to the Department of Physics
on June 22, 1994 in partial fulfillment
of the requirements for the Degree of
Doctor of Philosophy in Physics

Abstract

The research described in this thesis is divided into two parts. The first part concerns the calculation of quantum and thermal corrections to a supersymmetric Chern-Simons field theory. The second part studies Chern-Simons solitons in $2+1$ dimensional Einstein gravity.

In Chapter I we study radiative corrections to the Abelian self-dual Chern-Simons theory at zero and finite temperature. The analysis is performed with the help of functional methods. We consider the supersymmetric extension of scalar matter fields minimally coupled to a gauge field whose dynamics are governed solely by the Chern-Simons term. The scalar field potential is a self-dual sixth order polynomial with U(1)-symmetry-breaking and symmetry-preserving minima which are degenerate. We find that the zero-temperature one-loop radiative corrections do not remove this degeneracy and both minima remain supersymmetric. We calculate the leading-order finite-temperature contributions to the effective potential in the high-temperature limit and we find that the U(1)-symmetry is restored. In contrast to four-dimensional field theories that restore the U(1)-symmetry at high-temperature, the restoration of the U(1)-symmetry in the abelian self-dual Chern-Simons theory occurs at the two-loop level. The Chern-Simons system without supersymmetry is discussed, as well as the scalar field model without Chern-Simons gauge fields. The same finite temperature result emerges in these cases.

In Chapter II we consider the Abelian Chern-Simons-Higgs model in $2+1$ dimensional curved space. We obtain coupled nonlinear differential equations for the Einstein-scalar-gauge field equations. The equations are solved numerically to obtain topological soliton solutions. These solitons have mass and angular momentum. Numerical results show that space-time created by these solitons do not possess closed time-like curves, unlike that of spinning point particles.

Thesis Supervisor: Dr. Roman Jackiw

Title: Professor of Physics

Acknowledgements

I would like to express my gratitude to Professor Roman Jackiw for his guidance and patience and Professor John Negele for the support he has given. I wish to thank Miguel Ortiz without whose help second part of this thesis would not be accomplished. I thank Martin Leblanc and Teresa Thomaz for their collaboration in the first part of this thesis. I also would like to thank Professors Paul Joss and Alan Guth for several suggestions on this thesis. Special thanks goes to Professors Namık Kemal Pak and Metin Durgut for making me interested in Physics.

I would like to thank my fellow CTP graduate students, especially Eric Sather, Jim Olness, Laurent Lellouch, Niklas Dellby, Edi Halyo, Peter Unrau and Qiang Liu for their invaluable friendship.

Last but not the least I thank my parents Hikmet and Ayşen İpekoğlu, my aunt Zeliha İpekoğlu, my brother Mehmet and my sister Esma İpekoğlu for their unceasing love and support.

I gratefully acknowledge the financial support of the Scientific and Technical Research Council of Turkey during the first three years of my graduate studies.

Table of Contents

Introduction

Much effort has been devoted in the last decade to the study of planar physics. This interest has been mainly spurred by the introduction of Chern-Simons theory which has led to many interesting applications [1]. One reason for studying lower dimensional physics is pedagogical. One can study lower dimensional toy models with the hope to gain further insights and useful lessons to understand the physical four-dimensional world better. Another reason is that these lower dimensional models may actually be instrumental in describing real physical processes which are confined to a spatial plane. The Quantum Hall effect and high T_c superconductivity are prominent examples.

In this thesis we consider two such lower dimensional models, both of which have the Abelian Chern-Simons term as the sole kinetic term for the gauge field. In the first part we study the radiative corrections to a supersymmetric model [2] at both zero and finite temperature. This model is the supersymmetric extension of a system which has been found to have topological as well as nontopological soliton (vortex) solutions [3]. In the second part we study these solitons in curved space. We find numerical solutions.

Chern-Simons Solitons[1]

It is well known that Landau-Ginzburg macroscopic theory of superconductivity admits localized soliton (vortex) solutions [4]. The Abelian Higgs model, which is the relativistic extension of Landau-Ginzburg theory, has also been shown to have vortex solutions [5]. These vortices carry magnetic flux but no electric charge. But there are other possibilities. With the introduction of the Chern-Simons term in the Abelian Higgs model it was observed that there exist vortex solutions [6]. These vortices are different from the Nielsen-Olesen ones in that they carry electric charge as well as magnetic flux.

Another possibility is to have the Chern-Simons term only. This is not unreasonable since the Chern-Simons term dominates the higher derivative Maxwell term at large distances. The Lagrangian density is

$$\mathcal{L} = |D_\mu \phi|^2 + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - V(\phi) \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and the Minkowski space metric $\eta_{\mu\nu}$ is $(1, -1, -1)$, and the scalar field potential is

$$V(\phi) = \frac{e^4}{\kappa^2}|\phi|^2(|\phi|^2 - v^2)^2. \tag{2}$$

[1] The following is primarily taken from the references in [3]

The field equations are

$$D_\mu D^\mu \phi = -\frac{\partial V}{\partial \phi^*} \tag{3}$$

and

$$\frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = J^\alpha \tag{4}$$

where the conserved current $J^\mu = (\rho, \vec{J})$ is given by

$$J_\mu = ie(\phi^* D_\mu \phi - \phi D_\mu \phi^*). \tag{5}$$

The time component of the Eq. (3) is the Gauss' Law,

$$-\kappa B = \rho \tag{6}$$

where $B = -F^{12}$. This equation implies that any object that carries magnetic flux must also carry electric charge, and vice versa.

Time independent vortex solutions are stationary points of the energy functional which is

$$E = \int d^2x\{|D_0\phi|^2 + |\vec{D}\phi|^2 + V(\phi)\}. \tag{7}$$

After some manipulations and integration by parts one can obtain a lower bound for the energy (see references in [3] for details)

$$E \geq ev^2|\Phi| \tag{8}$$

where Φ is the magnetic flux. This lower bound is saturated by fields obeying the self-duality or Bogomol'nyi type equations

$$D_1\phi = \mp i D_2\phi \tag{9}$$

$$eB = \pm\frac{2e^2}{|\kappa|}|\phi|^2(1-\frac{|\phi|^2}{v^2}) \tag{10}$$

where the upper(lower) sign corresponds to positive(negative) values of Φ. Note that to satisfy self-duality equations the potential must be the special sixth order polynomial given in Eq. (2).

Solutions with axial symmetry corresponding to n elementary vortices superimposed at the origin can be written in the form

$$\phi(r,\varphi) = vR(r)e^{in\varphi}, \tag{11a}$$

$$A_\varphi(r,\varphi) = \frac{1}{e}[P(r)-n] \tag{11b}$$

$$A_r(r,\varphi) = 0. \tag{11c}$$

In order that the fields be nonsingular at the origin the boundary conditions $R(0) = 0$ and $P(0) = n$ must be imposed. For topological soliton solutions ϕ must approach to asymmetric vacuum at spatial infinity. This implies $R(\infty) = 1$. Finiteness of energy requires $P(\infty) = 0$. With the equations (11) the Eqs. (9) and (10) give

$$R' = \pm\frac{PR}{r} \tag{12}$$

$$\frac{P'}{r} = \pm \frac{2e^2v^2}{|\kappa|} R^2(R^2 - 1). \tag{13}$$

One can also obtain the following results for the flux and the angular momentum:

$$\Phi = \frac{2\pi n}{e} \tag{14}$$

$$J = -\frac{\pi\kappa n^2}{e^2} \tag{15}$$

The specific sixth order potential is necessary to achieve self-duality. It has also been shown that this potential arises naturally once a supersymmetric generalization is sought [2]. The fermionic part of this supersymmetric model is given as

$$\mathcal{L}_{\mathrm{F}} = i\bar{\psi}\gamma^\mu D_\mu \psi + \frac{e^2}{\kappa}(3|\phi|^2 - v^2)\bar{\psi}\psi. \tag{16}$$

The bosonic part is given by Eq. (1).

In the first part of the this thesis we calculate the radiative corrections to the potential in the supersymmetric model to see if the supersymmetry is broken. This analysis was done in collaboration with M. Leblanc and M. T. Thomaz and published in *Annals of Physics* (N.Y.) **214** (1992) 160.

Einstein Gravity in $2+1$ Dimensions

Einstein gravity in $2+1$ dimensions has attracted much attention recently. This is partly because it serves as a toy model to understand quantized gravity.

Another reason is that some four dimensional systems can be considered effectively three dimensional due to symmetry . In gravity this is true for the space-time created by an infinite cosmic string.

Cosmic strings are predicted to exist by some but not all grand unified theories. They are typically formed during phase transitions in the very early universe and are characterized by a mass scale of 10^{16} GeV. Cosmic strings have provided a hope for a satisfactory explanation of structure formation in the universe. They may be the source of primordial inhomogeneities in an otherwise homogeneous universe. There are two basic mechanisms for this. The first is the accretion of mass onto string loops which are produced when a segment of string crosses itself. Smaller loops can be the seed of galaxies and the larger ones of clusters of galaxies. The second mechanism involves long string segments. Velocity perturbations form in the wake of moving strings. This can be the source of large-scale structure by causing planar density perturbations. For further information on the cosmic strings and their role in the structure formation see Refs. [7] and the references therein.

In the following we review some basic facts about Einstein gravity in $2+1$ dimensions.

The Einstein's equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{17}$$

in any number of space-time dimensions. 2+1 dimensional space-time is peculiar in the sense that the Einstein tensor and the Riemann curvature tensor are equivalent [8]

$$R_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}G^{\rho\gamma}. \tag{18}$$

This makes three dimensional space-time dynamically trivial. Outside the matter sources the space-time is flat. This has two important consequences. The first is that there are no gravitational waves since all the vacuum solutions are trivial. Second, there are stable static solutions because matter in one region cannot affect objects in other regions. All effects of localized matter sources are on the global geometry. For example consider a particle of mass M and spin J at rest at the origin, which gives rise to the line element [8]

$$ds^2 = (dt + 4GJd\varphi)^2 - r^{-8GM}(dr^2 + r^2 d\varphi^2). \tag{19}$$

To see that the space-time is flat outside the origin we change to another coordinate system

$$\begin{aligned} t' &= t + 4GJ\varphi, \\ r' &= \frac{r^{1-4GM}}{1-4GM}, \\ \theta &= (1-4GM)\varphi. \end{aligned} \tag{20}$$

In this coordinate system the line element reads

$$ds^2 = dt'^2 - r'^2 d\theta^2 - dr'^2. \tag{21}$$

However these coordinates do not describe a globally Minkowskian space-time but one that has a conical geometry because θ ranges from $-(1-4GM)\pi$ to $(1-4GM)\pi$.

In general relativity there exists space-time solutions which admit closed time-like curves [9]. In $2+1$ dimensions the space-time of a spinning point particle admits closed time-like curves. Recently a number of articles have been published on the closed time-like curves in $2+1$ dimensions [10]. It is generally believed, however, that the laws of physics do not allow the appearance of closed time-like curves. This is known as the "Chronology Protection Conjecture"[11].

In the second part of this thesis we solve for the space-time created by Chern-Simons solitons that carry spin. We find that these extended particles do not create closed time-like curves.

References

[1] R. Jackiw and S. Templeton, Phys. Rev. **D23** (1981)2291

J. Schonfeld, Nucl. Phys. **B185** (1981)157

S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1982)975; Ann. Phys. (N.Y.) **140** (1982) 372

[2] C. Lee, K. Lee and E.J. Weinberg, Phys. Lett. **B243** (1990)105

[3] J. Hong, Y. Kim and P.Y. Pac, Phys. Rev. Lett. **64** (1990)2230

R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. **64** (1990)2234

R. Jackiw, K. Lee and E. J. Weinberg, Phys. Rev. **D42** (1990) 3488

[4] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20** (1950)1064

A. A. Abrikosov, Zh. Eksp. Teor. Fiz. **32** (1957)956 [Sov. Phys. JETP **5** (1957)1174]

[5] H. Nielsen and P. Olesen, Nucl. Phys. **B61** (1973)45

[6] S.Paul and A. Khare, Phys. Lett. **B174** (1986)420

L.Jacobs, A. Khare, C. Kumar and S. Paul, Int. J. Mod. Phys. **A6**

(1991)3441

[7] T. W. B. Kibble, J. Phys **A9** (1976) 1387; Phys. Rep. **67** (1980) 183;

A. Vilenkin, Phys. Rev. Lett. **46** (1981) 1169; 46 1496(E); Phys. Rep. **121** (1985) 263;

R. Brandenberger, Phys. Scripta **T36** (1991) 114; "Topological Defects and Structure Formation," BROWN-HET-906 May 1993;

T. Vachaspati, "Topological Defects in Cosmology," Lectures Delivered at ICTP, Trieste, July 1993;

A. Vilenkin and E. P. S. Shellard, "Cosmic Strings and Other Topological Defects," Cambridge University Press, 1994

[8] S. Deser, R. Jackiw and G. t 'Hooft, Ann. Phys. **152** (1984)220

[9] K. S. Thorne, in *Proceedings of the 13th International Conference on General Relativity and Gravitation*, ed. C. Kozameh (Institute of Physics, Bristol, England, 1993)

[10] J. R. Gott, Phys. Rev. Lett. **66** (1991)1126

S. Deser, R. Jackiw and G. t 'Hooft, Phys. Rev. Lett. **68** (1992)267

S. M. Carroll, E. Farhi and A. H. Guth, Phys. Rev. Lett. **68** (1992)263

D. Kabat, Phys. Rev. **D46** (1992)2720

[11] S. Hawking, Phys. Rev. **D46** (1992)603

I

Thermal and Quantum Fluctuations in Supersymmetric Chern-Simons Theory

In the last decade much interest has been directed toward the study of lower dimensional field theories. Some of this interest has been motivated by string theory, which necessitates understanding two-dimensional field theories. Others are interested in (2+1)-dimensional, planar gauge theories. The reasons for studying planar gauge theories are numerous and stem in part from the fact that interesting structures emerge such as the Chern-Simons term [1]. Such models may describe the quantum Hall effect and/or high-T_c-superconductivity (see ref.[2] and references therein).

The reason for the present work emerges from the recent investigations of R. Jackiw and E.J. Weinberg [3], and J.H. Hong, Y. Kim and P.Y. Pac [4] who studied vortex solutions in an Abelian Chern-Simons theory with spontaneous symmetry breaking. The system contains charged planar matter described by a scalar field interacting with a gauge field whose dynamics is governed solely by the Chern-

Simons term. When the scalar field potential supports U(1)-symmetry breaking, the model has one gauge degree of freedom together with the Higgs mode [5], while a U(1)-symmetric potential gives rise to two degrees of freedom associated with the scalar field. In the case where the theory is specialized to a specific ϕ^6-potential with two types of minima —U(1)-symmetry-breaking and -preserving— there exist time-independent charged topological vortex solutions to the field equations that approach the asymmetric vacuum at spatial infinity [3,4]. Furthermore, the symmetric vacuum admits charged non-topological soliton solutions [6]. Both types of solutions satisfy self-dual or Bogomol'nyi-type equations [7]. The specific sixth-order potential needed to obtain the self-dual equations arises naturally with a N=2-supersymmetric generalization of the bosonic system and both the U(1)-symmetry-breaking and symmetry-preserving minima are supersymmetric [8]. A natural question emerges at this point: Do the radiative corrections, including the finite temperature effects, lift the degeneracy of the self-dual potential? We analyze this quantum phenomenon for the supersymmetric self-dual system and find that the degeneracy is not lifted at one-loop for zero-temperature, however the U(1)-symmetry is restored in the high-temperature limit, a phenomenon that appears only after the inclusion of two-loop effects to the effective potential. Both minima remain supersymmetric at zero-temperature.

This chapter has the following structure: We begin in section II with a short discussion of the functional method for evaluating the effective potential, including the temperature dependent part. In section III, we apply the formalism at one-loop order to the N=2-supersymmetric Abelian self-dual Chern-Simons model [8]. We focus on two special cases a) zero-temperature and b) high-temperature. In section IV, we calculate the two-loop leading-order finite-temperature contributions to the effective potential. In section V, we show that it is sufficient to consider only the leading-order temperature dependence to the effective potential, that is higher-loop graphs (with more than two loops) do not contribute to the leading-order temperature dependence to the effective potential and hence are negligible. In section VI, we discuss the U(1)-symmetry restoration. Our conclusions are presented in section VII. In Appendix A, we collect the results of the integrals needed to evaluate the effective potential. In Appendix B, we analyse the Chern-Simons model without supersymmetry as well as the scalar field model without Chern-Simons gauge fields at zero-temperature. Both of these models exhibit U(1)-symmetry-restoration at high-temperature for the same reasons as the supersymmetric model.

Functional Evaluation of the Effective Potential

There are several methods of calculating the effective potential [9,10,11]. We use the functional method due to R. Jackiw [9]. Here we give a brief review of this method to establish the notation.

Consider a theory described by the lagrangian $\mathcal{L}$ depending on a set of fields $\phi_a(x)$:

$$I(\phi) = \int d^n x \ \mathcal{L}\{\phi_a(x)\} \quad , \tag{1}$$

where n is the dimension of the space-time. Next we define another Lagrangian by shifting the fields, $\phi_a(x) \to \varphi_a + \phi_a(x)$,

$$\begin{aligned} I(\varphi + \phi) - I(\varphi) - \int d^n x \ \phi_a(x) \frac{\delta I(\phi)}{\delta \phi_a(x)}\Big|_{\phi=\varphi} \\ \equiv \int d^n x \ \widetilde{\mathcal{L}}\{\varphi_a; \phi_a(x)\}, \end{aligned} \tag{2}$$

where the field φ_a is an x-independent quantity. We consider only scalar fields, but it is not difficult to generalize to the case when particles with higher spin are considered. The equation (2) can be rewritten into the form

$$\begin{aligned} \int d^n x \ \widetilde{\mathcal{L}}\{\varphi_a; \phi_a(x)\} = \int d^n x \ d^n y \ \frac{1}{2} \phi_a(x) i\mathcal{D}^{-1}_{ab}\{\varphi_a; x-y\} \phi_b(y) \\ + \int d^n x \ \widetilde{\mathcal{L}}_I\{\varphi_a; \phi_a(x)\} \end{aligned} \tag{3}$$

where the propagator $\mathcal{D}_{ab}$ satisfy

$$i\mathcal{D}^{-1}_{ab}\{\varphi_a; x-y\} = \frac{\delta^2 I(\phi)}{\delta \phi_a(x) \delta \phi_b(y)}\Big|_{\phi=\varphi} \quad , \tag{4}$$

and has a Fourier transformation given by[1]

$$i\mathcal{D}^{-1}_{ab}\{\varphi_a; k\} = \int d^n x \; e^{ikx} i\mathcal{D}^{-1}_{ab}\{\varphi_a; x\} \quad . \tag{5}$$

The effective action is defined as the Legendre transform of the connected generating functional and it is the generator of one-particle irreducible connected graphs. The effective potential is obtained from the effective action when the latter is evaluated for a constant field configuration $\Gamma(\varphi) = -V(\varphi)\int d^n x$ [9]. The formula for the effective potential $V(\varphi)$ is

$$\begin{aligned} V(\varphi) =& V_0(\varphi) - \frac{i\hbar}{2}\int \frac{d^n k}{(2\pi)^n} \ln det \;\; i\mathcal{D}^{-1}_{ab}\{\varphi_a; k\} \\ &+ i\hbar \langle T \exp\Big(\frac{i}{\hbar}\int d^n x \; \widetilde{\mathcal{L}}_I\{\varphi_a; \phi_a(x)\}\Big)\rangle_{1PI} \quad . \end{aligned} \tag{6}$$

The first term of this expression is the classical potential. The second term corresponds to the contribution of all graphs with one-loop. The determinant is over the indices a, b which can refer to internal or spin degrees of freedom. The last term summarizes contributions from higher-loop graphs. An over-all space-time volume factor has been deleted in the last term.

The finite temperature contribution to the effective potential can also be obtained by the functional method (we follow the discussion of Dolan and Jackiw [12]). It is possible to find the temperature dependence of the effective potential

[1] In our conventions the free-field spin-zero propagator is $\frac{i}{k^2 - m^2 + i\epsilon}$.

by using the same method as in the case of zero-temperature field theory with the crucial distinction that the free propagators of the theory satisfy the same differential equation but with a different time boundary condition [12,13].

As an example consider a self-interacting scalar field theory. We determine the finite-temperature 2-point function with the imaginary time, which is defined as

$$\mathcal{D}_\beta(x-y) = \frac{\mathrm{Tr}\ e^{-\beta H} T\phi(x)\phi(y)}{\mathrm{Tr}\ e^{-\beta H}}. \tag{7}$$

The time arguments of $\mathcal{D}_\beta$ are continued to $0 \leq ix_0,\ iy_0 \leq \beta$, and the Fourier inverse of (5) becomes

$$\mathcal{D}_\beta(x) = \int_k \ e^{-ikx}\mathcal{D}_\beta(k), \tag{8}$$

where $\int_k$ stands for $\frac{1}{(-i\beta)}\sum_{N=0,\pm1,\ldots}\int \frac{d^{n-1}k}{(2\pi)^{n-1}}$. The n-vector k has time component $\omega_N = \frac{2\pi N}{(-i\beta)}$ [2]. For non-interacting fields,

$$\begin{aligned}\mathcal{D}_\beta(k) &= \frac{i}{k^2 - m^2} \\ &= \frac{-i}{(4\pi^2 N^2/\beta^2) + \mathbf{k}^2 + m^2} \quad .\end{aligned} \tag{9}$$

The formula for the finite-temperature effective potential is again (6) with the substitution $\int \frac{d^n k}{(2\pi)^n} \to \int_k$ in the second term, and all higher-loop graphs are calculated from Wick contractions according to the last term of (6) but with the free 2-point function given by (9). We write the effective potential to $\mathcal{O}(\hbar)$ as

$$V^\beta(\varphi) = V_0 + V_1^\beta \tag{10}$$

[2] In the case of a fermion field, the time component reads $\omega_N = \frac{\pi(2N+1)}{(-i\beta)}$.

where

$$V_1^\beta = V_1^0 + \overline{V}_1^\beta, \tag{11}$$

and V_0 is the classical potential and is temperature independent. The second term V_1^β can be written in two parts, V_1^0 that is the zero-temperature contribution to the effective potential to $\mathcal{O}(\hbar)$ and $\overline{V}_1^\beta$ that contains all the temperature dependence to $\mathcal{O}(\hbar)$. From (6), we have

$$V_1^\beta = -\frac{i\hbar}{2} \int_k \ln \ \det \ i\mathcal{D}_{ab}^{-1}\{\varphi_a; k\} \quad . \tag{12}$$

One-Loop Calculations

The bosonic Abelian self-dual Chern-Simons theory has been discussed in the classical regime by R. Jackiw and E.J. Weinberg [3], and by J. Hong, Y. Kim and P.Y. Pac [4]. The supersymmetric theory then was constructed so that the self-dual equations emerge as a consequence of supersymmetry [8]. We present here a calculation of the one-loop contribution to the scalar field potential in the N=2-supersymmetric Abelian model. We consider the following Lagrangian[3]

$$\begin{aligned}\mathcal{L} =& \frac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + |D_\mu\phi|^2 + i\overline{\psi}\gamma^\mu D_\mu\psi \\ &-(\frac{e^2}{\kappa})^2|\phi|^2(|\phi|^2 - v^2)^2 + (\frac{e^2}{\kappa})(3|\phi|^2 - v^2)\overline{\psi}\psi \end{aligned} \tag{13}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and the charged scalar field $\phi(x)$ can be written in terms of two real scalar fields as $\phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)]$.

We shift the fields by a constant and define a new Lagrangian following (2). Since we are seeking the corrections to the scalar field potential, it is unnecessary to shift the A_μ and ψ fields. To one-loop order, it is sufficient to consider the quadratic part of the shifted Lagrangian. We find that the quadratic part of the shifted Lagrangian is

$$\begin{aligned}&\int d^3x\ \widetilde{\mathcal{L}}_0\{\varphi_a;\phi_a(x), A_\mu(x), \overline{\psi}(x), \psi(x)\} \\ &= \int d^3x\ d^3x' \Big[\frac{1}{2}\phi_a(x)i\mathcal{D}^{-1}_{ab}\{\varphi_a; x-x'\}\phi_b(x') + \overline{\psi}(x)iS^{-1}\{\varphi_a; x-x'\}\psi(x') \\ &\quad + \frac{1}{2}A^\mu(x)i\Delta^{-1}_{\mu\nu}\{\varphi_a; x-x'\}A^\nu(x') + A^\mu(x)M_{\mu a}\{\varphi_a; x-x'\}\phi_a(x')\Big]\end{aligned} \tag{14}$$

[3] We use the metric $g^{\mu\nu} = \mathrm{diag}(1,-1,-1)$ and our γ-matrices obey $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\alpha}\gamma_\alpha$.

where

$$
\begin{aligned}
i\mathcal{D}^{-1}_{ab}\{\varphi_a; x-x'\} &= \Big[(-\Box - m_1^2)\delta_{ab} - \frac{m_2^2}{2}\frac{\varphi_a\varphi_b}{\rho^2}\Big]\delta^3(x-x') \\
iS^{-1}\{\varphi_a; x-x'\} &= \big(i\,\partial\!\!\!/ - m_f\big)\delta^3(x-x') \\
i\Delta^{-1}_{\mu\nu}\{\varphi_a; x-x'\} &= \big(\kappa\epsilon_{\mu\lambda\nu}\partial^\lambda + 2e^2\rho^2 g_{\mu\nu} + \frac{1}{\alpha}\partial_\mu\partial_\nu\big)\delta^3(x-x') \\
M_{\mu a}\{\varphi_a; x-x'\} &= -e\epsilon_{ab}\varphi_b\partial_\mu\delta^3(x-x') \quad .
\end{aligned}
\qquad (15)
$$

Here a, b run from 1 to 2, and $\rho^2 \equiv \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$ where φ_1 and φ_2 are real x-independent fields. The class of Lorentz gauges $\mathcal{L}_{G.F.} = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$ have been used for the gauge fixing . The parameters m_1^2, m_2^2 and m_f present in (15) are given by

$$
\begin{aligned}
m_1^2 &= \frac{1}{2\rho}\frac{dV(\rho)}{d\rho} = \frac{e^4}{\kappa^2}\big[(\rho^2 - v^2)(3\rho^2 - v^2)\big] \\
m_2^2 &= \frac{1}{2}\Big[\frac{d^2V}{d\rho^2} - \frac{1}{\rho}\frac{dV}{d\rho}\Big] = \frac{4e^4}{\kappa^2}\big[\rho^2(3\rho^2 - 2v^2)\big] \\
m_f &= G(\rho) = \frac{e^2}{\kappa}(v^2 - 3\rho^2) \quad .
\end{aligned}
\qquad (16)
$$

The first equality defines the parameters m_1^2, m_2^2 and m_f in terms of an arbitrary scalar field potential $V(\rho)$, and an arbitrary fermionic potential $G(\rho)\overline{\psi}\psi$ for a general Lagrangian. The second equality corresponds to the particular case where the potentials are those given in (13), that is, $V(\rho) = \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2$ and $G(\rho) = \frac{e^2}{\kappa}(v^2 - 3\rho^2)$. In what follows we shall express our results in terms of the parameters m_1^2, m_2^2 and m_f which can be deduced for arbitrary functions $V(\rho)$ and $G(\rho)$.

When the parameters m_1^2, m_2^2 and m_f are evaluated at the classical degenerate minima, they provide the particle mass spectrum. In the symmetric vacuum $\rho = 0$, we find that the theory (13) contains two scalar-field degrees of freedom both with masses $m_s = \frac{e^2v^2}{|\kappa|}$. The fermions are also massive with $m_f = m_s$. The two scalars and the fermions correspond to the four degrees of freedom of the theory. For the asymmetric vacuum, $\rho = v$, the scalar field sector develops one Goldstone boson and one massive Higgs. The mass of the Higgs is $m_H = 2\frac{e^2v^2}{|\kappa|}$. The fermions are also massive and again $|m_f| = m_H$. The Goldstone boson combines with the Chern-Simons gauge field: the mass of the Chern-Simons gauge field is $m_A = |m_f| = m_H$. Four degrees of freedom are present in the asymmetric vacuum [3,5].

The effective potential to the $\mathcal{O}(\hbar)$ is given by (10-12). To arrive at this result in the case where more than scalar fields are considered requires a few steps that are described now. In the present case, fermions do not couple to the other fields and they are easily integrated in the functional integral. To this order the bosonic fields enter in quadratic form, however due to the presence of the coupling $A_\mu M_a^\mu \phi_a$ a standard change of variable for the scalar field is necessary in order to decouple the A_μ-field from the scalar field ϕ [9]. Once the change of variable is made, it is possible to evaluate the path integral and the effective potential reduces in the momentum space representation to

$$\begin{aligned} V^{\beta}(\varphi) = V_0 &- \frac{i\hbar}{2}\int_k \ln\ \det(i\mathcal{D}^{-1}_{ab}\{\varphi_a;k\}) \\ &- \frac{i\hbar}{2}\int_k \ln\ \det(i\Delta^{-1}_{\mu\nu}\{\varphi_a;k\} + iN_{\mu\nu}\{\varphi_a;k\}) \\ &+ i\hbar \int_k \ln\ \det(iS^{-1}\{\varphi_a;k\}), \end{aligned} \tag{17}$$

where the determinant is over the scalar field internal space indices a, b, the spinor indices and the Lorentz indices μ,ν [9]. The presence of the matrix $N_{\mu\nu}\{\varphi_a;k\} = M_{\mu a}\{\varphi_a;k\}\mathcal{D}_{ab}\{\varphi_a;k\}M_{\nu b}\{\varphi_a;-k\}$ is due to the gauge field-scalar transition. After inverting $\mathcal{D}^{-1}_{ab}\{\varphi_a;k\}$ to get the expression $\mathcal{D}_{ab}\{\varphi_a;k\} = \frac{i}{k^2-m_1^2-m_2^2+i\epsilon}\frac{\varphi_a\varphi_b}{2\rho^2} + \frac{i}{k^2-m_1^2+i\epsilon}(\delta_{ab} - \frac{\varphi_a\varphi_b}{2\rho^2})$, it is straightforward to obtain

$$N_{\mu\nu}\{\varphi_a;k\} = \frac{2ie^2\rho^2 k_\mu k_\nu}{k^2 - m_1^2 + i\epsilon} \quad . \tag{18}$$

The momentum representation for the propagators is obtained by making the substitution $\partial_\mu \to -ik_\mu$ in (15). An evaluation of the determinants in (17) leads to

$$\begin{aligned} V^{\beta} = V_0 &- \frac{i\hbar}{2}\int_k \ln\bigl(k^2 - (m_1^2+m_2^2) + i\epsilon\bigr) + \ln(k^2 - m_1^2 + i\epsilon) \\ &- \frac{i\hbar}{2}\int_k \ln(k^4 - m_1^2k^2 + 2\alpha e^2\rho^2 m_1^2) - \ln(k^2 - m_1^2 + i\epsilon) \\ &- \frac{i\hbar}{2}\int_k \ln\bigl(k^2 - 4\frac{e^4\rho^4}{\kappa^2} + i\epsilon\bigr) \\ &+ i\hbar\int_k \ln(k^2 - m_f^2) \quad . \end{aligned} \tag{19}$$

This expression is valid for arbitrary α. The first integral comes from the scalar fields, the second and third integrals come from the gauge fields and the gauge field-scalar transition. The last term of (19) is due to the presence of fermions. We note from (16), for the specific sixth-order potential, that the parameters m_1^2 and $m_1^2 + m_2^2$ become negative for $\frac{1}{3} < \frac{\rho^2}{v^2} < 1$ and $\frac{1}{5}[2 - \sqrt{\frac{7}{3}}] \approx 0.09 < \frac{\rho^2}{v^2} < \frac{1}{5}[2 + \sqrt{\frac{7}{3}}] \approx 0.71$ respectively. In the rest of this paper we consider only the Landau gauge $\alpha = 0$.

The integrals given in (19) contains the temperature dependence of the effective potential. It is of interest to decompose (19) into two parts, the zero-temperature and the temperature dependent contributions as in (11). To see this, we note that all integrals in (19) are of the form $U_1^\beta = -\frac{i\hbar}{2}\int_k \ln(k^2 - M^2 + i\epsilon)$, $U_1^\beta = I(M^2, \beta)$ for bosons and $U_1^\beta = F(M^2, \beta)$ for fermions. We evaluate these integrals in Appendix A [12]. We obtain

$$\begin{aligned} U_1^\beta(M^2) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2}\left[\frac{E_M}{2} + \frac{1}{\beta}\ln(1 \pm e^{-\beta E_M})\right] \\ &= U_1^0(M^2) + \overline{U}_1^\beta(M^2) \end{aligned} \tag{20a}$$

with

$$U_1^0(M^2) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\,\frac{E_M}{2}, \tag{20b}$$

$$\overline{U}_1^\beta(M^2) = \frac{1}{2\pi\beta^3}\int_0^\infty dx\; x\ln\left(1 \pm e^{-\sqrt{x^2 + M^2\beta^2}}\right) \tag{20c}$$

where the upper (lower) sign correspond to fermions (bosons) contribution, $E_M^2 = \mathbf{k}^2 + M^2$, $\beta = \frac{1}{k_B\,T}$, and k_B is the Boltzman constant. That U_1^0 is the usual form of the zero-temperature effective potential [9] where $U_1^0(M^2) = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\ \ln(-k_0^2 + \mathbf{k}^2 + M^2 - i\epsilon)$ comes from the fact that [12], apart from an infinite constant, $-\frac{i}{2}\int \frac{dk_0}{(2\pi)}\ \ln(-k_0^2 + E^2 - i\epsilon) = \frac{1}{2}E$.

Using (20a-c), we rewrite the effective potential (19) as

$$V^\beta = V_0(\rho) + V_1^0(\rho) + \overline{V}_1^\beta(\rho) \tag{21a}$$

where

$$V_1^0(\rho) = \frac{1}{2}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\Big[E_{(m_1^2+m_2^2)^{1/2}} + E_{m_1} + E_{\frac{2e^2\rho^2}{|\kappa|}} - 2E_{m_f}\Big] \tag{21b}$$

and

$$\begin{aligned}\overline{V}_1^\beta(\rho) =& \frac{1}{2\pi\beta^3}\int_0^\infty dx\ x\Big[\ln\Big(1 - e^{-\sqrt{x^2+[m_1^2+m_2^2]\beta^2}}\Big) + \ln\Big(1 - e^{-\sqrt{x^2+m_1^2\beta^2}}\Big) \\ &+ \ln\Big(1 - e^{-\sqrt{x^2+4\frac{e^4\rho^4}{\kappa^2}\beta^2}}\Big) - 2\ln\Big(1 + e^{-\sqrt{x^2+m_f^2\beta^2}}\Big)\Big].\end{aligned} \tag{21c}$$

It is possible to evaluate (20b) for M^2 of arbitrary sign

$$U_1^0(M) = \frac{\Lambda}{4\pi^2}M^2 - \frac{M^3}{12\pi} \tag{22}$$

where $M^3 \equiv -i|M|^3$ for negative M^2.

The finite-temperature contributions (21c) vanish as they should at zero temperature, $\beta \to \infty$. The expression (21c) is not reliable for the values of ρ for which

$m_1^2+m_2^2$ and m_1^2 become negative and the effective potential becomes complex [9]. We calculate (see Appendix A) the finite-temperature contribution to the effective potential as an expansion around $a^2 \equiv M^2\beta^2 \approx 0$:

$$\begin{aligned}\overline{V}_1^{\beta}(M^2) =& \sum_{M^2}\Big[-\frac{\zeta(3)}{2\pi\beta^3} - \frac{M^2}{8\pi\beta}\big(\ln\beta^2 + \ln M^2 - \sqrt{M^2\beta^2}\big) + \mathcal{O}(\beta M^4)\Big] \\ &+ \Big[-\frac{3\zeta(3)}{4\pi\beta^3} + \frac{m_f^2}{2\pi\beta}\big(\ln 2 - \frac{1}{2}\sqrt{m_f^2\beta^2}\big) + \mathcal{O}(\beta m_f^4)\Big] \qquad (23)\end{aligned}$$

where by the sum we add the three contributions of (21c) with M^2 taking the values $m_1^2+m_2^2$, m_1^2 and $4\frac{e^4\rho^4}{\kappa^2}$. The $\frac{1}{\beta^3}$ and the $\frac{\ln\beta}{\beta}$ terms in the sum in (23) do not become complex for negative M^2, hence we may rely on them. The contribution from fermions to (23) are also reliable, however the field-dependent terms are of lower order in β. We conclude that fermions do not contribute to the one-loop leading-order finite-temperature of the effective potential. The reason for the difference between the leading-order finite-temperature coming from bosons and fermions has to do with the infrared properties of the integrals encountered in each case. The logarithmic temperature dependence from bosons has its origin in the vanishing of one of its energy-eigenvalues and in the fact that the leading divergence in the one-loop n-point expansion is linear.

We find for the unrenormalized effective potential (the temperature dependent term is just the high-temperature contribution)

$$V^{\beta}(\rho) = \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2 + \frac{\hbar e^4\Lambda}{\pi^2\kappa^2}\rho^2(\rho^2 - v^2)$$

$$
\begin{aligned}
&- \frac{\hbar e^6}{12\pi|\kappa|^3}\Big[(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2} \\
&\qquad + 8\rho^6 - 2|3\rho^2 - v^2|^3\Big] \\
&- \frac{\hbar e^4}{\pi\kappa^2}\Big[(\frac{11}{2}\rho^4 - 4v^2\rho^2)\frac{\ln\beta}{\beta}\Big].
\end{aligned} \tag{24}
$$

Λ is our cut-off for ultra-violet divergences. The Λ-dependence appears only in the $T = 0$ portion as expected. We also removed the term $\frac{\zeta(3)}{\beta^3}$ since it is ρ-independent. The values for m_1^2, m_2^2, m_f have been substituted from (16).

We now renormalize. Since the theory is renormalizable, we need only a finite number of parameter to absorb the infinity coming from the Λ-dependence ($\Lambda \to \infty$). We find that it is possible to absorb (to $\mathcal{O}(\hbar)$) the Λ-dependence in just one parameter. We define a new parameter $v_r^2 = v^2 - \frac{\hbar\Lambda}{2\pi^2}$. In this way, v_r^2 is arbitrary and V^β is Λ-independent therefore finite. Owing to the supersymmetry the radiative corrections are calculable. We rewrite the renormalized potential (24) with the substitution $v^2 \to v_r^2$, and then drop the subscript "r",

$$
\begin{aligned}
V^\beta(\rho) =&\frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2 \\
&- \frac{\hbar e^6}{12\pi|\kappa|^3}\Big[(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2} \\
&\qquad + 8\rho^6 - 2|3\rho^2 - v^2|^3\Big] \\
&- \frac{\hbar e^4}{\pi\kappa^2}\Big[(\frac{11}{2}\rho^4 - 4v^2\rho^2)\frac{\ln\beta}{\beta}\Big] \quad .
\end{aligned} \tag{25}
$$

We note that a minimal subtraction scheme has been used here without any reference to the normalization condition imposed on the coupling constant of the theory.

There are two cases that we pursue: a) zero-temperature and b) high-temperature limit.

A. The Zero-Temperature One-Loop Corrections

We analyze the result (25) at zero temperature. In this case, the effective potential can be written as

$$V(\rho) = V_0(\rho) + \lambda V_1(\rho), \tag{26}$$

where $\lambda = \frac{\hbar e^2}{|\kappa|}$,

$$V_0(\rho) = \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2, \tag{27}$$

$$\begin{aligned} V_1(\rho) = -\frac{e^4}{12\pi\kappa^2}\Big[&(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2} \\ &+ 8\rho^6 - 2|3\rho^2 - v^2|^3\Big]. \end{aligned} \tag{28}$$

Since the radiative corrections to the classical potential are first order in λ we can write each new minimum as the sum of the corresponding classical minimum and a first order term in λ, i.e.,

$$\rho_i^{\text{new}} = \rho_i + \lambda\delta\rho_i \,, \qquad i = 1, 2 \tag{29}$$

where $\rho_1 = 0$ and $\rho_2 = v$. After a little algebra it is easy to show that

$$\delta\rho_i = -\frac{dV_1/d\rho}{d^2V_0/d\rho^2}\bigg|_{\rho=\rho_i}. \tag{30}$$

One can ask if the radiative corrections remove the degeneracy of the classical minima and therefore break the supersymmetry. It has been proven for certain supersymmetric models that if the supersymmetry is not broken at the tree level it cannot be broken perturbatively by quantum corrections [14]. We find that this holds true for our case. To see this one has to evaluate the effective potential to $\mathcal{O}(\lambda)$ at the new minima:

$$\begin{aligned} V(\rho_i + \lambda\delta\rho_i) &= V_0(\rho_i + \lambda\delta\rho_i) + \lambda V_1(\rho_i + \lambda\delta\rho_i) \\ &= V_0(\rho_i) + \lambda\ \delta\rho_i \frac{dV_0}{d\rho}\bigg|_{\rho=\rho_i} + \lambda V_1(\rho_i) + \mathcal{O}(\lambda^2)\ \ . \end{aligned} \tag{31}$$

The first two terms are obviously zero for the classical minima. It is also easy to check from (28) that

$$V_1(\rho = 0) = V_1(\rho = v) = 0. \tag{32}$$

Hence the degeneracy of the minima is not removed and both minima remain supersymmetric to $\mathcal{O}(\lambda)$.

B. The High-Temperature One-Loop Effects

We study the complete expression (25). We have two parameters that control the behavior of the effective potential in the high-temperature limit; the parameter $\lambda = \frac{\hbar e^2}{|\kappa|}$ controls radiative corrections and β provides the temperature dependence. (25) is evaluated under the condition $\beta \ll v^{-2}$. However, the expansion parameter used for fixed β is $a^2 = M^2\beta^2 \ll 1$ with M^2 being either of $m_1^2 + m_2^2$, m_1^2, m_f^2, or $4\frac{e^4\rho^4}{\kappa^2}$. Combining the condition $\beta \ll v^{-2}$ with $a^2 \ll 1$ gives a constraint on the value ρ can take for large ρ in order that the expansion in a^2 of the effective potential be reliable.

In the case where the high-temperature limit is considered at one-loop, the last term on the R.H.S of (25) dominates the others. Therefore, the potential (25) can be well approximated by

$$V^{\beta}_{H.T.}(\rho) = -\frac{\hbar e^4}{\pi\kappa^2}\left(\frac{11}{4}\rho^4 - 4v^2\rho^2\right)\frac{\ln\beta}{\beta} \quad . \tag{33}$$

In the high-temperature limit, we see that $\rho = 0$ becomes a relative maximum of the effective potential, and that the absolute minimum is located at $\rho = \sqrt{\frac{8}{11}}v \approx 0.85v$. This is true for the values $\beta < 1$ where $\ln\beta < 0$.

As the temperature increases, it is clear that at one-loop the symmetry-breaking form of the potential remains. However, we cannot conclude that this remains true to all orders, the analysis of the temperature dependence of the effective potential requires higher-loop consideration –a feature that is different than for 4-dimensional field theories– since we are considering a ϕ^6-potential.

Two-Loop Calculations

We consider the leading-order high-temperature effects to two-loop order to the scalar-field effective potential for the supersymmetric self-dual Chern-Simons theory. We shall discuss in the next section that it is sufficient to consider only the two-loop order contributions to obtain the leading-order temperature dependence.

The two-loop contributions arise from Wick contractions according to the last term of (6). To evaluate the last term of (6), we need to find $\tilde{\mathcal{L}}_I$, the interaction part of the Lagrangian defined as (2). To do this, we shift the scalar field by a constant. The quadratic part of the shifted Lagrangian has been found to be (14). The interaction part is

$$\begin{aligned}
\tilde{\mathcal{L}}_I = {} & eA_\mu(x)\epsilon_{ab}\phi_a(x)\partial^\mu\phi_b(x) + e^2\varphi_a\phi_a(x)A_\mu(x)A^\mu(x) - eA_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x) \\
& - \frac{1}{8}(\frac{e^2}{\kappa})^2\Big[(12\varphi^2 - 16v^2)\varphi_a\phi_a(x)\phi^2(x) + 8\varphi_a\varphi_b\varphi_c\phi_a(x)\phi_b(x)\phi_c(x)\Big] \\
& + \frac{3e^2}{\kappa}\varphi_a\phi_a(x)\overline{\psi}(x)\psi(x) + \frac{e^2}{2}\phi^2(x)A_\mu(x)A^\mu(x) + \frac{3e^2}{2\kappa}\phi^2(x)\overline{\psi}(x)\psi(x) \\
& - \frac{1}{8}(\frac{e^2}{\kappa})^2\Big[(3\varphi^2 - 4v^2)\phi^4(x) + 12\varphi_a\varphi_b\phi_a(x)\phi_b(x)\phi^2(x)\Big] \\
& + \mathcal{O}(\varphi_a\phi_a\phi^4(x)) + \mathcal{O}(\phi^6(x)).
\end{aligned} \tag{34}$$

As was mentioned above (17), a standard shift of variable $\phi_a(x) \to \phi_a(x) + ie\epsilon_{bc}\varphi_c \int \mathcal{D}_{ab}(x,y)\partial_\mu A^\mu(y)d^4y$ is needed in order to decouple the A_μ-field from the scalar field. In this way, we do not encounter any gauge-scalar field transition.

Under this change of variable the interaction part of the Lagrangian changes, however, in the Landau gauge all the "new" part of the interaction Lagrangian do not contribute to the effective action. It is sufficient to consider (34) as the interaction Lagrangian with diagonal propagators given by

$$\langle T\phi_a(x)\phi_b(y)\rangle = \int_p e^{-ip(x-y)}\Big[\frac{i\frac{\varphi_a\varphi_b}{2\rho^2}}{p^2-m_1^2-m_2^2+i\epsilon}+\frac{i(\delta_{ab}-\frac{\varphi_a\varphi_b}{2\rho^2})}{p^2-m_1^2+i\epsilon}\Big],$$
$$\langle TA_\mu(x)A_\nu(y)\rangle = \int_p e^{-ip(x-y)}\frac{-i}{\kappa^2p^2-4e^4\rho^4+i\epsilon}[2e^2\rho^2(g_{\mu\nu}-p_\mu p_\nu/p^2)+i\kappa\epsilon_{\mu\tau\nu}p^\tau],$$
$$\langle T\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle = \int_p e^{-ip(x-y)}\frac{i(\not p+m_f)_{\alpha\beta}}{p^2-m_f^2+i\epsilon} \quad . \tag{35}$$

We now calculate the 1PI leading-order temperature dependent contribution to the effective potential following (6). The two-loop effective potential is given by

$$V_2^\beta(\rho) = i\langle Ti\int d^3x\tilde{\mathcal{L}}_I(x)\rangle + \frac{i}{2!}\langle T(i\int d^3x\tilde{\mathcal{L}}_I(x)\ \ i\int d^3y\tilde{\mathcal{L}}_I(y))\rangle\ . \tag{36}$$

We are interested only in the leading-order temperature dependent part of the effective potential. The first term in (36) corresponds to those graphs presented in Fig.1. They are the Wick contraction of the four-particle interaction vertices. Their evaluation is straightforward since they are products of single loops. We obtain a leading-order temperature dependence from Fig. 1a,b

$$i\langle Ti\int d^3x\tilde{\mathcal{L}}_I(x)\rangle = \frac{e^4}{\kappa^2}\Big(1+\frac{9}{2}\Big)\Big(\frac{\hbar\ln\beta}{\pi\beta}\Big)^2\rho^2 = \frac{11e^4}{2\kappa^2}\Big(\frac{\hbar\ln\beta}{\pi\beta}\Big)^2\rho^2 \quad . \tag{37}$$

Fig.1c does not contribute because, as a consequence of (23), fermions do not contribute to the leading-order temperature dependence.

The second term in (36) are the overlapping divergent graphs. They are presented in Fig.2. They arise from Wick contractions of three-particle interaction vertices. To simplify the discussion of the evaluation of the graphs in Fig.2, we present a discussion on the behavior of a one-loop graph following Weinberg [15].

Consider a single loop constructed from boson propagators in three-space-time dimension with superficial divergence D. For simplicity, set the external momentum to be zero. We can rescale the internal momenta as well as energies by a factor β^{-1}, so that the n-point one-loop graph takes the form

$$\beta^{-D} I(m_{int}\beta) \ , \qquad D = 3 - 2n, \tag{38}$$

where m_{int} represents the various internal masses. Thus for $\beta \to 0$, the loop behaves like β^{-D}, unless there are infrared divergences when the arguments of the function I vanish. In three-space-time dimension, $D \leq 1$ for all one-loop graphs and therefore all graphs are affected by such infrared-divergences, and occur for the case where the internal lines of the loop represent a boson with zero-energy. For $D = 1$ (the tadpole), the infrared-divergence goes as $\ln \beta$. For $D < 1$, the infrared-divergence goes as β^{D-1}, and in this case the n-point ($n \neq 1$) one-loop graph behave no worse than β^{-1}. We conclude that the leading-order temperature dependence for any one-loop graph in three-space-time dimension arises for bosonic fields whenever there is a linearly ultraviolet-divergent integral.

We now present the evaluation of Fig. 2a. The interaction Lagrangian part that give rise to this graph is $e^2\varphi_a\phi_a(x)A_\mu(x)A^\mu(x)$ and the graph reads after Wick contraction of the fields

$$\frac{-4e^4\rho^2}{\kappa^2}\int_p\Big[\frac{1}{p^2-m_1^2-m_2^2+i\epsilon}\Big]\int_l\frac{(l-p)^\mu l_\mu}{((l-p)^2-4e^4\rho^4/\kappa^2)(l^2-4e^4\rho^4/\kappa^2)}. \qquad (39)$$

Using Feynman's parameters and an appropriate shift in the l-momentum we can rewrite the integral over l as

$$\frac{1}{-i\beta}\sum_N\int\frac{d^2\mathbf{l}'}{(2\pi)^2}\int_0^1 dx$$
$$\frac{(l'-px)_\mu(l'+p(1-x))_\mu}{[-4\pi^2\beta^2(N-M(1-x))^2-\mathbf{l}'^2-(4e^4\rho^4/\kappa^2-p^2x(1-x))]^2} \qquad (40)$$

where $l'_\mu = l_\mu - p_\mu(1-x)$. There are two types of integrals in (40): one that is linearly UV-divergent on the three dimensional momentum space and one that is convergent. We have discussed the behavior of one-loop n-point functions above and we concluded that only the linearly UV-divergent one-loop boson graphs are to be taken into consideration. For the linearly UV-divergent integral, after rescaling the internal momenta as well as the energy, we set $N = M = 0$ where the infrared-divergence occurs. We find for (40)

$$\frac{i}{2\pi\beta}\ln\beta \quad . \qquad (41)$$

Combining this result with the leading-order temperature dependence coming from the p-integration in (39) lead us to the result for Fig. 2a

$$\frac{e^4}{\kappa^2}\Big(\frac{\hbar\ln\beta}{\pi\beta}\Big)^2\rho^2 \quad . \qquad (42)$$

To present the final two-loop result, we discuss the graphs of Fig. 2b–e. Only the graphs composed of product of linearly UV-divergent one-loop boson subgraphs can provide the leading-order temperature dependence to the effective potential. The fermions will not contribute to the leading-order temperature dependence to the effective potential since fermion-loop integral never carry zero-energy fermions –a fermion-loop integral can increase no faster than β^{-1} as $\beta \to 0$– hence, the graphs of Fig. 2d–e do not contribute. By naive power counting, the graphs of Fig. 2b–c are less than quadratically divergent which means that it is made of one-loop subgraphs with at least one of the subgraph being less than linearly divergent. Therefore the Fig. 2b–c do not contribute to the leading-order temperature dependence to the effective potential.

The leading-order temperature dependence of the two-loop effective potential is

$$V^{\beta}(\rho) = \frac{e^4}{\kappa^2}\Big[\rho^2(\rho^2 - v^2)^2 - (\frac{11}{2}\rho^4 - 4v^2\rho^2)(\frac{\hbar \ln \beta}{\pi\beta}) + \frac{13}{2}\rho^2(\frac{\hbar \ln \beta}{\pi\beta})^2\Big] \quad . \qquad (43)$$

High Temperature Higher Loop Effects

We mentioned at the beginning of the previous section that it would be sufficient to consider only up to two-loop contributions in order to obtain the leading-order temperature dependence to the effective potential. We show this now.

Consider the path-integral for the scalar field part of the Lagrangian (13)

$$\int [d\phi] exp\Big\{ \frac{i}{\hbar} \int \big[|\partial_\mu \phi|^2 - \frac{e^4}{\kappa^2} |\phi|^2 (|\phi|^2 - v^2)^2 \big] \Big\} \quad . \tag{44}$$

The discussion including the gauge field and the fermion field is the same. Rescale the scalar field by $\Phi^2 = \frac{e^2}{|\kappa|}|\phi|^2$ and the classical expectation value of the field by $\omega^2 = \frac{e^2}{|\kappa|}v^2$. The path-integral (44) becomes (up to a constant in the functional measure)

$$\int [d\Phi] exp\Big\{ \frac{i|\kappa|}{e^2 \hbar} \int \big[|\partial_\mu \Phi|^2 - \Phi^2 (\Phi^2 - \omega^2)^2 \big] \Big\} \quad . \tag{45}$$

The path-integral (45) shows that the expansion parameter of the theory is $\lambda = \frac{\hbar e^2}{|\kappa|}$ which follows the loop counting, and $\lambda \ll 1$. Φ^2, $|\phi|^2$, ω^2 and β^{-1} have dimension of a mass. From dimensional analysis and (45), we obtain the field and temperature structure for the perturbative series in the high-temperature limit to be

$$\begin{aligned} V^\beta =& \Phi^2(\Phi^2 - \omega^2)^2 + (a\Phi^4 + b\omega^2\Phi^2)(\lambda \frac{\ln \beta}{\beta}) + c\Phi^2(\lambda \frac{\ln\beta}{\beta})^2 \\ &+ d(\lambda\frac{\ln\beta}{\beta})^3 + (e\Phi^4 + f\omega^2\Phi^2)(\frac{\lambda}{\beta}) + g\Phi^2(\lambda\frac{\ln\beta}{\beta})(\frac{\lambda}{\beta}) \\ &+ h(\frac{\lambda\ln\beta}{\beta})^2(\frac{\lambda}{\beta}) \quad . \end{aligned} \tag{46}$$

We do not include terms of the form $V^{\beta}_{p,q} \approx \lambda^p(\lambda\frac{\ln\beta}{\beta})^q$, and those terms of next to the leading-order in β to $V^{\beta}_{p,q}$, where p and q are integers and $p \neq 0$. These terms are suppressed by the coupling constant λ. The constants a to h are dimensionless. The terms with the constants e to h are those terms of next to the leading-order temperature dependence, hence we drop them. The term with the constant d is the contribution that arises from a three-loop evaluation of the leading-order temperature dependence of the effective potential. Figure 3 shows examples. It is field-independent and therefore can be dropped from the effective potential. Indeed, any higher-loop graphs (with the number of loops$\geq$ 3) will be either field-independent or next and lower to the leading-order temperature dependence. It can be understood also from the fact that higher-loop graphs become more convergent in a renormalizable theory. Only terms with the constants a to c contribute in the leading-order approximation to the effective potential in the high-temperature limit and they are those that we have evaluated in the previous sections.

Now using the Lagrangian of (13) and the same argument leading to (46) we obtain the same field and temperature behavior as the potential (46). Converting to the field $\rho^2 = \frac{\varphi^2}{2}$ we conclude that the form of the effective potential in the leading-order temperature dependence is

$$V^{\beta}(\rho) = \frac{e^4}{\kappa^2}\left[\rho^2(\rho^2 - v^2)^2 + (a\rho^4 + bv^2\rho^2)(\hbar\frac{\ln\beta}{\beta}) + c\rho^2(\hbar\frac{\ln\beta}{\beta})^2\right]$$

$$= \frac{e^4}{\kappa^2}\Big[\rho^6 + e(\beta)\rho^4 + m^2(\beta)\rho^2\Big] \tag{47}$$

where $e(\beta) = (a\frac{\hbar \ln\beta}{\beta} - 2v^2)$, $m^2(\beta) = [c(\frac{\hbar\ln\beta}{\beta})^2 - bv^2\frac{\hbar\ln\beta}{\beta} + v^4]$ and $a = -\frac{11}{2\pi}$, $b = \frac{4}{\pi}$, and $c = \frac{13}{2\pi^2}$.

Discussion of the U(1)-Symmetry Restoration

A sufficient condition for U(1)-symmetry restoration is that the symmetric vacuum becomes the true vacuum. It is clear that the U(1)-symmetry is restored at high-temperature because the two-loop effect is positive and dominates all the other contributions in such a limit. In fact, since $m^2(\beta)$ is positive for all β, the slope of the effective potential remains positive for any value of ρ as long as the coefficient $e(\beta)$ of (47) remains positive, in other words, as long as $\frac{-\hbar \ln \beta}{\beta} \geq \frac{4\pi v^2}{11} \approx 1.14\, v^2$. In the region where the value of β is such that $\frac{-\hbar \ln \beta}{\beta} \approx 1.14\, v^2$ but with $e(\beta) < 0$, we can still say that the U(1)-symmetry is restored. However, for large values of β ($T \rightarrow 0$) our approximation scheme is invalid and we cannot make any statement about the U(1)-symmetry restoration in the low-temperature regime.

Conclusions

We have demonstrated that the zero-temperature one-loop radiative corrections to the effective potential do not lift the degeneracy of the classical potential. Both minima remain supersymmetric. The supersymmetry ensures that it is sufficient to have only one parameter "v" to perform the renormalization. We have also shown that the symmetry is restored at high-temperature; however, in contrast to the U(1)-symmetry restoration of four-dimensional field theories, the phenomenon appears from a two-loop calculation. Graphs with more than two loops do not contribute to the leading-order temperature dependence.

Appendix A

Evaluation of Integrals

We evaluate the integrals encountered in (20). The integral for bosons is

$$\begin{aligned} I(M^2,\beta) &= -\frac{i}{2}\int_k \ln(k^2 - M^2 + i\epsilon) \\ &= \frac{1}{2\beta}\sum_{n=-\infty}^{n=+\infty}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\ln\Big(\frac{4\pi^2 n^2}{\beta^2} + E_M^2\Big) \end{aligned} \tag{A.1a}$$

and for fermions is

$$\begin{aligned} F(M^2,\beta) &= -\frac{i}{2}\int_k \ln(k^2 - M^2 + i\epsilon) \\ &= \frac{1}{2\beta}\sum_{n=-\infty}^{n=+\infty}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\ln\Big(\frac{\pi^2(2n+1)^2}{\beta^2} + E_M^2\Big) \end{aligned} \tag{A.1b}$$

where $E_M^2 = \mathbf{k}^2 + M^2$. We start with the evaluation of (A.1a). Following Dolan and Jackiw in reference [12] (eqs. (22-24)), we find that

$$I(M^2,\beta) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\Big[\frac{E_M}{2} + \frac{1}{\beta}\ln(1 - e^{-\beta E_M})\Big]. \tag{A.2}$$

We now perform the zero-temperature contribution integral $I(M^2, \beta = \infty)$

$$\begin{aligned} I(M^2,\beta=\infty) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{E_M}{2} \\ &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\ln(-k_0^2 + E_M^2 - i\epsilon) \end{aligned} \tag{A.3}$$

where we dropped divergent quantities that are M-independent [12] in order to write down the second equality. The measure d^3k is $dk_0 dk_1 dk_2$. Upon performing a Wick rotation on (A.3), we find the Euclidean integral

$$\begin{aligned} I(M^2, \beta = \infty) &= \frac{1}{(2\pi)^2} \int_0^\Lambda dx\ x^2 \ln(x^2 + M^2) \\ &= \frac{M^2 \Lambda}{4\pi^2} - \frac{M^3}{12\pi} \quad . \end{aligned} \tag{A.4}$$

We have dropped in the last equality the M-independent terms and terms that vanish when $\Lambda \to \infty$. In the case where $M^2 < 0$, a contour integration in the complex plane is sufficient to evaluate the integral, and we obtain

$$I(M^2, \beta = \infty) = \frac{\Lambda M^2}{4\pi^2} + \frac{i}{12\pi} |M|^3. \tag{A.5}$$

We now turn to the evaluation of the temperature dependent integral in (A.2) (see also (20c)). The integral is

$$\begin{aligned} I(\beta, a^2) &= \frac{1}{\beta} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \ln\left(1 - e^{-\beta\sqrt{\mathbf{k}^2 + M^2}}\right) \\ &= \frac{1}{2\pi\beta^3} \int_0^\infty dx\ x \ln\left(1 - e^{-\sqrt{x^2 + a^2}}\right), \end{aligned} \tag{A.6}$$

where the parameter $a^2 = M^2\beta^2$. We now follow the discussion of Dolan and Jackiw [12]. In the $a^2 \ll 1$ limit, (A.6) can be written as the following expansion

$$I(\beta; a^2) \approx I(\beta; 0) + a^2 \frac{\partial I}{\partial a^2}(\beta; a^2) + \cdots \tag{A.7}$$

It is easy to evaluate the first integral in (A.7), and we get $I(\beta,0) = -\frac{1}{2\pi\beta^3}\zeta(3)$. The second term can be obtained by differentiating $I(\beta,a^2)$ once,

$$\frac{\partial I(\beta;a^2)}{\partial a^2} = \frac{1}{4\pi\beta^3}\int_0^\infty dx\ \frac{x}{\sqrt{x^2+a^2}}\frac{1}{e^{\sqrt{x^2+a^2}}-1} \quad . \tag{A.8}$$

We note that (A.8) is finite for any values of a^2.

To evaluate (A.8), we define the regulated expression

$$A_\epsilon = \int_0^\infty dx\ \frac{x^{1-\epsilon}}{\sqrt{x^2+a^2}}\frac{1}{e^{\sqrt{x^2+a^2}}-1} \quad . \tag{A.9}$$

and $\lim_{\epsilon\to 0}\frac{1}{4\pi\beta^3}A_\epsilon = \frac{\partial I}{\partial a^2}(\beta;a^2)$.

A_ϵ can be written in terms of the sum of two expressions: $A_\epsilon^{(1)} = \int_0^\infty dx x^{1-\epsilon}\sum_{n=-\infty}^{\infty}\frac{1}{4\pi^2n^2+a^2+x^2}$, and $A_\epsilon^{(2)} = -\frac{1}{2}\int_0^\infty dx\frac{x^{1-\epsilon}}{\sqrt{x^2+a^2}}$. The second of those integral is easy to perform with the help of the beta-function [16]

$$B(m,n) = \int_0^\infty dt\frac{t^{m-1}}{(1+t)^{m+n}} \quad , \tag{A.10}$$

and we obtain

$$A_\epsilon^{(2)}(a^2) = \frac{1}{2}\sqrt{a^2} + \mathcal{O}(\epsilon). \tag{A.11}$$

The first integral is more intricate and we present some of the steps of how to perform the integration. By making a change of variable $y = \frac{x}{\sqrt{4\pi^2n^2+a^2}}$, $A_\epsilon^{(1)}$ is written as

$$\begin{aligned} A_\epsilon^{(1)}(a^2) &= \sum_{n=-\infty}^{\infty}(4\pi^2n^2+a^2)^{-\epsilon/2}\int_0^\infty dy\frac{y^{1-\epsilon}}{1+y^2} \\ &= \frac{1}{2}\frac{\pi}{\sin(\pi\epsilon/2)}\sum_{n=-\infty}^{\infty}(a^2+4\pi^2n^2)^{-\epsilon/2} \end{aligned} \tag{A.12}$$

where in going from the first to the second equality the definition (A.10) for the beta-function, the relations $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ and $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(\pi\epsilon/2)}$ have been used [16].

After some algebraic manipulations, we rewrite (A.12) as

$$\begin{aligned} A_\epsilon^{(1)}(a^2) =& \frac{\pi}{2}\frac{1}{\sin(\epsilon\pi/2)}\Big[(a^2)^{-\epsilon/2} + 2\sum_{n=1}^{\infty}(4\pi^2n^2)^{-\epsilon/2} \\ &+ 2\sum_{n=1}^{\infty}(4\pi^2n^2)^{-\epsilon/2}\Big((1+\frac{a^2}{4\pi^2n^2})^{-\epsilon/2} - 1\Big)\Big] \\ =& \frac{1}{\epsilon}\Big\{-\frac{\epsilon}{2}\ln a^2 + \sum_{n=1}^{\infty}(4\pi^2n^2)^{-\epsilon/2}\times \\ & \Big[(-\epsilon)\ln\Big(1+\frac{a^2}{4\pi^2n^2}\Big)\Big] + \mathcal{O}(\epsilon^2)\Big\} \quad . \end{aligned} \tag{A.13}$$

To go from the first equality to the second, we made an expansion around $\epsilon \approx 0$, for this reason the first term provides the $\ln a^2$-term. We regularized the second term in the first equality with the use of the zeta-function, $\sum_{n=1}^{\infty}(4\pi^2n^2)^{-\epsilon/2} = \frac{\zeta(\epsilon)}{(2\pi)^\epsilon}$ and $\zeta(\epsilon) = -\frac{1}{2} - \frac{\epsilon}{2}\ln(2\pi) + \mathcal{O}(\epsilon^2)$. It is possible to show that the last term on the R.H.S. of the second equality of (A.13) reduces to

$$\sum_{n=1}^{\infty}(4\pi^2n^2)^{-\epsilon/2}(-\epsilon)\ln\Big(1+\frac{a^2}{4\pi^2n^2}\Big) = (-\epsilon)\Big[\ln\Big(\sinh\Big(\frac{\sqrt{a^2}}{2}\Big)\Big) - \ln\Big(\frac{\sqrt{a^2}}{2}\Big)\Big] + \mathcal{O}(\epsilon^2) \tag{A.14}$$

which enable us to write the final expression for $A_\epsilon^{(1)}$

$$A_\epsilon^{(1)}(a^2) = -\frac{1}{2}\Big\{\ln a^2 + \Big[\ln\Big(\sinh\Big(\frac{\sqrt{a^2}}{2}\Big)\Big) - \ln\Big(\frac{\sqrt{a^2}}{2}\Big)\Big]\Big\} + \mathcal{O}(\epsilon). \tag{A.15}$$

Combining $A_\epsilon^{(1)}$ with $A_\epsilon^{(2)}$, multiplying this sum by $\frac{1}{4\pi\beta^3}$, adding the result to $I(\beta;0)$, taking the limit $a^2 \approx 0$ and $\epsilon \to 0$ gives us the desired result

$$\frac{1}{\beta}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\ln\Big(1 - e^{-\beta\sqrt{\mathbf{k}^2+M^2}}\Big) \approx -\frac{\zeta(3)}{2\pi\beta^3} - \frac{a^2}{8\pi\beta^3}\left[\ln a^2 - \sqrt{a^2} + \mathcal{O}(a^2)\right] \quad (A.16)$$

with $a^2 = M^2\beta^2$. From (A.4) and (A.16) we find that (A.1a) is given keeping only the leading-order temperature dependence by

$$I(M^2,\beta) = \frac{M^2\Lambda}{4\pi^2} - \frac{M^3}{12\pi} - \frac{\zeta(3)}{2\pi\beta^3} - \frac{M^2}{4\pi\beta}\ln\beta \quad . \quad (A.17)$$

We now evaluate the integral (A.1b) which arises from the fermionic contribution. We can write $F(M^2,\beta)$ as [12]

$$F(M^2,\beta) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\Big[\frac{E_M}{2} + \frac{1}{\beta}\ln(1 + e^{-\beta E_M})\Big], \quad (A.18)$$

which has the same zero-temperature contribution (A.4) as the boson. The evaluation of the temperature dependence of $F(M^2,\beta)$ is similar to the boson case and we do not repeat it here. The result is

$$F(M^2,\beta) = \frac{M^2\Lambda}{4\pi^2} - \frac{M^3}{12\pi} + \frac{3\zeta(3)}{8\pi\beta^3} - \frac{M^2}{4\pi\beta}(\ln 2 - \frac{1}{2}\sqrt{M^2\beta^2}) + \mathcal{O}(\beta M^4). \quad (A.19)$$

The crucial distinction between (A.17) and (A.19) is that the M^2-dependent leading-order temperature dependence in (A.19) does not carry the logarithm-dependence on β. We find that this distinction between bosons and fermions is due to the presence of the vanishing energy-eigenvalue $\omega_{N=0} = 0$ for bosons [15]. Fermions do not have vanishing energy-eigenvalue; $\omega_N = \frac{(2N+1)\pi}{(-i\beta)}$ where $N = 0, \pm 1, \ldots$.

Appendix B

Abelian self-dual Chern-Simons theory

and scalar self-dual theory at zero-temperature.

We consider in this appendix the model without supersymmetry at zero-temperature. The Lagrangian for the Abelian self-dual Chern-Simons model is given by

$$\mathcal{L} = \frac{1}{4}\kappa\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + |D_\mu\phi|^2 - (\frac{e^2}{\kappa})^2|\phi|^2(|\phi|^2 - v^2)^2 \quad . \qquad (B.1)$$

Following the method presented in this paper we calculate the effective potential which is given by (19) without the fermionic contributions at $T = 0$ (with $\alpha = 0$).

The unrenormalized potential at one-loop is

$$\begin{aligned} V^\beta(\rho) =& \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2 + \frac{\hbar e^4\Lambda}{\pi^2\kappa^2}(\frac{11}{2}\rho^4 - 4v^2\rho^2) \\ & - \frac{\hbar e^6}{12\pi|\kappa|^3}\Big[(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} \\ & + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2} + 8\rho^6\Big] \quad . \qquad (B.2) \end{aligned}$$

Since this theory is renormalizable, we need only a finite number of parameter to absorb the infinity coming from the Λ-dependence ($\Lambda \to \infty$). In contrast to the supersymmetric case, we find that it is necessary to have two parameters to renormalize the model: v^2 and δm. To perform the renormalization in the minimal subtraction scheme, we define the new parameters $\delta m_r = \delta m - \frac{\hbar\Lambda}{\pi^2}\frac{19e^4v^2}{2\kappa^2}$, and $v_r^2 = v^2 + \frac{\hbar\Lambda}{\pi^2}\frac{11}{4}$.

We rewrite the renormalized potential (B.2) using the above definition and then drop the subscript "r", we obtain

$$\begin{aligned} V^\beta(\rho) =& \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2 + \delta m\ \rho^2 \\ & - \frac{\hbar e^6}{12\pi|\kappa|^3}\Big[(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} \\ & \qquad + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2} + 8\rho^6\Big] \quad . \end{aligned} \qquad (B.3)$$

The introduction of the second parameter δm needed to complete the renormalization brings an arbitrariness. The shape of the potential will depend upon the value δm. This is in contrast to the supersymmetric theory where only v^2 had to be renormalized. Therefore, without any specification on the values δm takes, it is impossible to determine whether or not we have a true or "false" vacuum at $\rho = 0$. The region for which the effective potential becomes complex is the same as in the supersymmetric case.

We finally consider the case where the Chern-Simons gauge fields are eliminated from the system. We consider the Lagrangian

$$\mathcal{L} = |\partial_\mu\phi|^2 - (\frac{e^2}{\kappa})^2|\phi|^2(|\phi|^2 - v^2)^2 \qquad (B.5)$$

and the effective potential contains only the first two terms of (19) at $T = 0$

The renormalized effective potential is given by

$$
\begin{aligned}
V^{\beta}(\rho) =& \frac{e^4}{\kappa^2}\rho^2(\rho^2 - v^2)^2 + \delta m\, \rho^2 \\
& - \frac{\hbar e^6}{12\pi|\kappa|^3}\Big[(15\rho^4 - 12v^2\rho^2 + v^4)^{3/2} \\
& \qquad + (3\rho^4 - 4v^2\rho^2 + v^4)^{3/2}\Big]
\end{aligned}
\qquad (B.6)
$$

where the renormalization parameters has been defined in a similar way as in the above case. Again, here, we cannot decide if the potential has a true or “false” vacuum at $\rho = 0$ when $T = 0$.

Both models discussed in this appendix restores the U(1)-symmetry in the high-temperature limit. The restoration of the U(1)-symmetry also occurs at the two-loop level. This result remains true for higher-order graphs.

References

[1] R. Jackiw and S. Templeton, Phys. Rev. **D23** (1981) 2291;

J. Schonfeld, Nucl. Phys. **B185** (1981) 157;

S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48** (1982) 975; Ann. Phys. (N.Y.) **140** (1982) 372.

[2] R. Jackiw, Phys. Rev. **D29** (1984) 2375, and Comm. Nucl. Particle Phys. **8** (1984) 15;

Y. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys.**B3** (1989) 1001.

[3] R. Jackiw, and E.J. Weinberg, Phys. Rev. Lett. **64** (1990)2234;

[4] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64** (1990)2230.

[5] R.D. Pisarski, and S. Rao, Phys. Rev. **D32**, (1985)2081;

S.K. Paul, and A. Khare, Phys. Lett. **B171** (1986)244;

S. Deser, and Z. Yang, Mod. Phys. Lett. **A4** (1989)2123.

[6] R. Jackiw, K. Lee, and E.J. Weinberg, Phys. Rev. **D42** (1990) 3488;

A. Khare, Bhubaneswar preprint IP-BBSR/90 (March 1990).

[7] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. **24** (1976)449 (Yad. Fiz. **24** (1976)861).

[8] C. Lee, K. Lee, and E.J. Weinberg, Phys. Lett. **B243** (1990)105.

[9] R. Jackiw, Phys. Rev. **D9** (1974)1686.

[10] S. Coleman, and E. Weinberg, Phys. Rev. **D7** (1973)1888.

[11] S. Weinberg, Phys. Rev. **D7** (1973) 2887.

[12] L. Dolan and R. Jackiw, Phys. Rev. **D9** (1974) 3320.

[13] C.W. Bernard, Phys. Rev. **D9** (1974) 3312.

[14] P. West, Nucl. Phys. **B106** (1976) 219;

D. Capper and M. Ramon Medrano, J. Phys. **62** (1976) 269;

S. Weinberg, Phys. Lett. **62B** (1976) 111.

[15] S. Weinberg, Phys. Rev. **D9** (1974)3357.

[16] M. Abramowitz, and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 9th edition (1970).

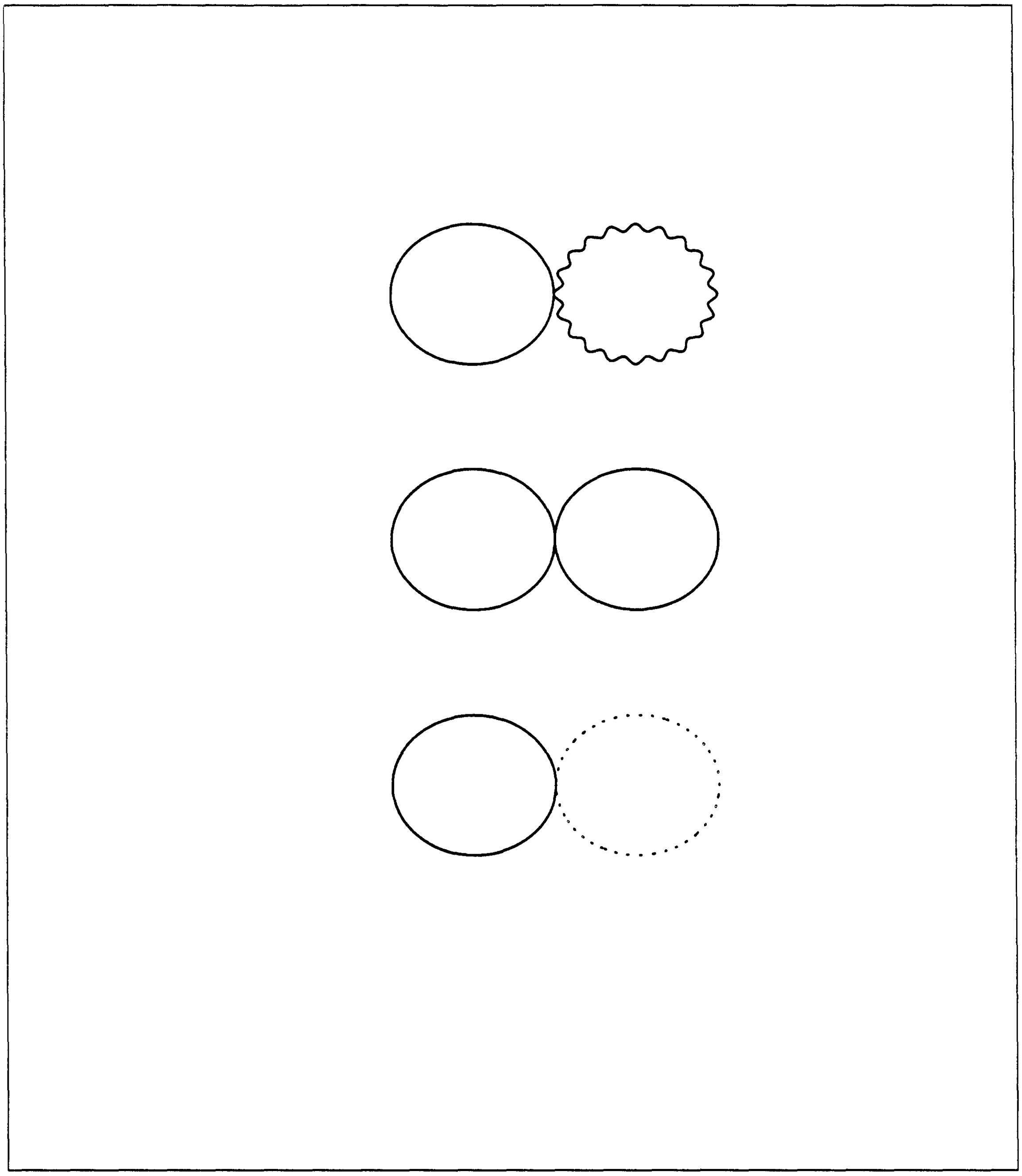

Figure 1. Two-loop graphs coming from Wick contractions of four-particle interaction vertices. (The continuous line represents the scalar field, the wavy line the vector field and the dashed line the fermion field.)

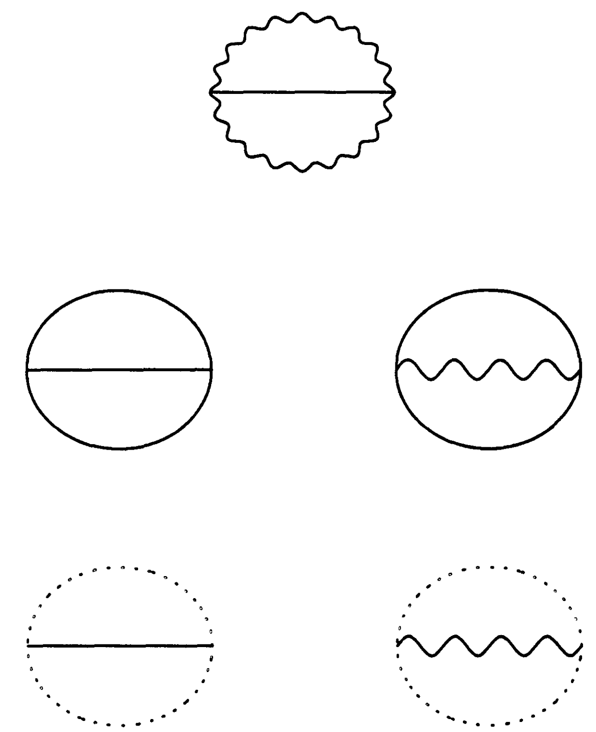

Figure 2. Two-loop graphs coming from Wick contractions of three-particle interaction vertices. (The conventions are the same as in Fig. 1.)

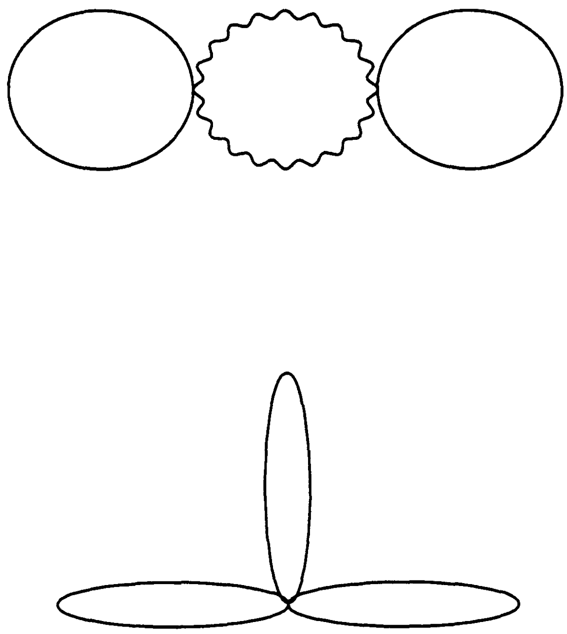

Figure 3. Sample three-loop graphs with high-temperature behavior $(\frac{\lambda \ln \beta}{\beta})^3$. They are both φ-independent. (The conventions are the same as in Fig. 1.)

II

Abelian Chern-Simons Solitons in Einstein Gravity

The Model

We consider an Abelian gauge theory in 2+1 space-time dimensions where the kinetic action for the gauge field A_μ consists of the Chern-Simons term only. The matter part comprise a charged scalar field ϕ with a sixth order potential which has a $U(1)$-symmetric as well as an asymmetric minimum that are degenerate. In terms of these fields the matter Lagrangian density in flat Minkowski space reads

$$\mathcal{L}_{\mathrm{M}} = (D_\mu\phi)^* D^\mu\phi + \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha - V(\phi) \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$, and the potential is

$$V(\phi) = \frac{e^4}{\kappa^2}|\phi|^2(|\phi|^2 - v^2)^2. \tag{2}$$

It has been shown that this particular choice of potential gives rise to self-dual topological solitons in flat space as well non-topological ones [1] [2] [3].

We are interested in finding the topological soliton solutions in three dimensional Einstein gravity, that is, solutions that approach to the asymmetric vacuum at spatial infinity.

The action in the curved space is given as the sum of the matter and the gravitational actions

$$S = S_{\rm M} + S_{\rm G}, \tag{3}$$

where

$$S_{\rm M} = \int d^3x\sqrt{g(x)}[g^{\mu\nu}(D_\mu\phi)^* D_\nu\phi + \frac{\epsilon^{\mu\nu\alpha}}{\sqrt{g(x)}}F_{\mu\nu}A_\alpha - V(\phi)] \tag{4}$$

and

$$S_{\rm G} = -\frac{1}{16\pi G}\int d^3x\sqrt{g(x)}R(x) \tag{5}$$

are the matter and the gravitational actions, respectively. $R(x)$ is the Ricci scalar, $g(x)$ is the determinant of the metric $g_{\mu\nu}(x)$ and G is the gravitational constant.

Variation of the total action S with respect to ϕ, A_μ and $g_{\mu\nu}$ yields the scalar and the gauge field equations

$$D_\mu(\sqrt{g}g^{\mu\nu}D_\nu\phi) + \sqrt{g}\frac{e^4}{\kappa^2}\phi(|\phi|^2 - v^2)(3|\phi|^2 - v^2) = 0, \tag{6}$$

$$\frac{\kappa}{2}\epsilon^{\alpha\mu\nu}F_{\mu\nu} + ie\sqrt{g}g^{\alpha\mu}[\phi(D_\mu\phi)^* - \phi^* D_\mu\phi] = 0, \tag{7}$$

and the Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{8}$$

where the energy-momentum tensor is [3]

$$T_{\mu\nu} = (D_\mu\phi)^* D_\nu\phi + D_\mu\phi(D_\nu\phi)^* - g_{\mu\nu}[(D_\alpha\phi)^* D^\alpha\phi - V(\phi)]. \tag{9}$$

Rotationally Symmetric and Stationary Solutions

In order to obtain solutions we will make further assumptions. We will assume that the fields are rotationally symmetric and take the following form:

$$\phi = vR(r)e^{in\varphi}, \tag{10a}$$

$$A_\varphi = \frac{1}{e}[P(r) - n], \tag{10b}$$

$$A_t = \frac{Q(r)}{e}, \tag{10c}$$

$$A_r = 0, \tag{10d}$$

where the integer n is the winding number. In addition the metric will be assumed to be stationary[1]

$$ds^2 = \lambda_{ab}(r)dx^a dx^b - dr^2 \tag{11}$$

where Latin letters a, b represent t and φ. The quantities λ_{ab} are functions of the radial coordinate r only. We can think of λ_{ab} as the metric of a two-dimensional subspace.

[1] A stationary, rotationally symmetric three-dimensional space-time has two commuting Killing vectors: $(\partial/\partial t)$ which is time-like, and $(\partial/\partial\varphi)$ which is space-like with closed orbits. One can therefore arrive at the metric (11) by choosing the coordinates appropriately. See, for example, refs. [4] and [5].

With these assumptions the scalar and the gauge field equations gives us the following system of nonlinear differential equations

$$\frac{1}{\sqrt{-\lambda}}\frac{d}{dr}(\sqrt{-\lambda}\frac{dR}{dr}) + \lambda^{tt}Q^2R + \lambda^{t\varphi}QPR$$
$$+ \lambda^{\varphi\varphi}P^2R - \beta^2 R(R^2-1)(3R^2-1) = 0, \qquad (12)$$

$$\frac{1}{\sqrt{-\lambda}}\frac{dP}{dr} + 2\beta R^2(\lambda^{tt}Q + \lambda^{t\varphi}P) = 0, \qquad (13)$$

$$\frac{1}{\sqrt{-\lambda}}\frac{dQ}{dr} - 2\beta R^2(\lambda^{t\varphi}Q + \lambda^{\varphi\varphi}P) = 0. \qquad (14)$$

We note that λ^{ab} is the inverse of the 2×2 matrix λ_{ab} and $\lambda \equiv \det(\lambda_{ab}) = -g$. We also define $\beta^2 \equiv \frac{e^4 v^4}{\kappa^2}$.

To obtain the Einstein's equations we introduce the notation [6] [7]

$$\chi^a_{\ b} = \lambda^{ac}\frac{d\lambda_{cb}}{dr}. \qquad (15)$$

These quantities form a two-dimensional tensor. With this notation the Einstein's equations can be written as

$$R^a_{\ b} = \frac{1}{2\sqrt{-\lambda}}\frac{d}{dr}(\sqrt{-\lambda}\chi^a_{\ b}) = 8\pi G(T^a_{\ b} - T\delta^a_{\ b}) \qquad (16)$$

and

$$G^r_{\ r} = -\frac{1}{4}\det(\chi^a_{\ b}) = 8\pi G T^r_{\ r}, \qquad (17)$$

where $R^\mu_{\ \nu}$ is the Ricci tensor and $T = T^\mu_{\ \mu}$ is the trace of the energy-momentum tensor.

Eqs. (16) and (17) give five coupled equations. This is clearly an overdetermined system since the metric has only three independent components. Using the symmetry of the Ricci tensor $R_{\mu\nu}$, however, we can show that one of the Eqs. (16), namely the equation for $R^{t}{}_{\varphi}$ (or R^{φ}_{t}), can be written in terms of the others, i.e.,

$$R^{t}{}_{\varphi} = \frac{1}{g_{tt}}[g_{\varphi\varphi}R^{\varphi}_{t} + g_{t\varphi}(R^{t}{}_{t} - R^{\varphi}{}_{\varphi})]. \tag{18}$$

One can also show that any solution that satisfies eqs.(16) also satisfies eq. (17) [7]. Hence we have only three independent equations from the Einstein's equations.

The components of the energy-momentum tensor are

$$T^{t}_{t} = v^2[(R')^2 + \lambda^{tt}Q^2R^2 - \lambda^{\varphi\varphi}P^2R^2 + \beta^2R^2(R^2-1)^2], \tag{19}$$

$$T^{\varphi}_{\varphi} = v^2[(R')^2 - \lambda^{tt}Q^2R^2 + \lambda^{\varphi\varphi}P^2R^2 + \beta^2R^2(R^2-1)^2], \tag{20}$$

$$T^{r}_{r} = v^2[-(R')^2 - \lambda^{tt}Q^2R^2 - 2\lambda^{t\varphi}PQR^2 - \lambda^{\varphi\varphi}P^2R^2 + \beta^2R^2(R^2-1)^2], \tag{21}$$

$$T^{t}_{\varphi} = 2v^2[\lambda^{tt}PQR^2 + \lambda^{t\varphi}P^2R^2], \tag{22}$$

$$T^{\varphi}_{t} = 2v^2[\lambda^{t\varphi}Q^2R^2 + \lambda^{\varphi\varphi}PQR^2], \tag{23}$$

and the trace T is

$$T = v^2[(R')^2 - \lambda^{tt}Q^2R^2 - 2\lambda^{t\varphi}PQR^2 - \lambda^{\varphi\varphi}P^2R^2 + \beta^2R^2(R^2-1)^2] \tag{24}$$

where ($'$) denotes $\frac{d}{dr}$.

We now choose the following parametrization for the stationary metric

$$ds^2 = e^{A(r)}(dt + K(r)d\varphi)^2 - e^{D(r)}d\varphi^2 - dr^2. \tag{25}$$

For convenience let us also introduce the definitions

$$\alpha \equiv 8\pi G v^2 \tag{26}$$

$$H(r) \equiv e^{\frac{A+D}{2}} = \sqrt{g}. \tag{27}$$

In terms of these definitions and the parametrization we obtain the following six coupled nonlinear differential equations from the Einstein-scalar-gauge field equations:

$$R'' + \frac{H'}{H}R' = -e^{-A}Q^2R + \frac{e^A}{H^2}R(KQ-P)^2 + \beta^2 R(R^2-1)(3R^2-1), \tag{28}$$

$$P' = -2\beta R^2[e^{-A}HQ - \frac{e^A}{H}K(KQ-P)], \tag{29}$$

$$Q' = 2\beta\frac{e^A}{H}R^2(KQ-P), \tag{30}$$

$$A'' + \frac{H'}{H}A' = -\frac{e^{2A}}{H^2}(K')^2 + 4\alpha R^2[e^{-A}Q^2 - \beta^2(R^2-1)^2], \tag{31}$$

$$K'' + (2A' - \frac{H'}{H})K' = -4\alpha e^{-A}QR^2(KQ-P), \tag{32}$$

$$H'' = -2\alpha HR^2[-e^{-A}Q^2 + \frac{e^A}{H^2}(KQ-P)^2 + 2\beta^2(R^2-1)^2]. \tag{33}$$

These six equations form the basis of our numerical analysis.

Boundary Conditions

In this section we establish the boundary conditions for the Eqs. (28)–(33). We then briefly describe the numerical method used.

We need a total of ten boundary conditions. We are searching for topological soliton solutions, i.e., solutions with non-zero winding number n. The boundary conditions at the origin for $R(r)$ and $P(r)$ follow from the requirement that the fields be nonsingular. This implies

$$P(0) = n, \tag{34}$$

and

$$R(0) = 0. \tag{35}$$

Finiteness of energy will require that the scalar field to approach to its value at one of the two vacua. This condition for the topological solitons becomes

$$R(\infty) = 1\,. \tag{36}$$

It will also be required that $P(\infty) = Q(\infty) = 0$.[2]

[2] One can work out the behaviour of $P(r)$ and $Q(r)$ at infinity and find that

$$P(r) \sim P_\infty \sqrt{r} e^{-2\beta r}$$
$$Q(r) \sim \frac{P_\infty}{\sqrt{r}} e^{-2\beta r}$$

where P_∞ is some constant. Therefore we need to require only one to approach zero at infinity as the boundary condition. The other will follow automatically. For our purposes we will not need the detailed behaviour of the functions at infinity therefore we will not expound this point.

We now determine the boundary conditions for the metric functions. The metric at the origin should approach to flat space metric expressed in a rotating coordinate system in the limit $r \to 0$

$$ds^2 \sim A_0 dt^2 + 2K_0 r^2 dt d\varphi - r^2 d\varphi^2 - dr^2. \tag{37}$$

We choose the normalization for the Killing vector $(\partial/\partial t)$ as 1. This requires $A_0 = 1$ which gives us the boundary condition

$$A(0) = 0. \tag{38}$$

Next by comparing Eqs. (25) and (37) we see that

$$K(r) \to K_0 r^2 \qquad \text{as} \qquad r \to 0. \tag{39}$$

where K_0 is an unknown constant. This behavior allows us to choose the following boundary conditions

$$K(0) = 0, \tag{40a}$$

$$K'(0) = 0. \tag{40b}$$

We see again by comparing Eqs. (25) and (37) that

$$e^{\frac{D}{2}} \to r \tag{41a}$$

and

$$(e^{\frac{D}{2}})' \to 1 \tag{41b}$$

as $r \to 0$. From this we obtain the boundary conditions on $H(r)$ as

$$H(0) = 0, \tag{42a}$$

$$H'(0) = 1. \tag{42b}$$

We need one more boundary condition for A(r). To obtain it we first observe that as $r \to 0$

$$H \to re^{\frac{A}{2}}, \tag{43a}$$

$$K' \to 2K_0 r, \tag{43b}$$

$$R \to r^n. \tag{43c}$$

Thus Eq.(31) can be rewritten near the origin as

$$(e^{\frac{A}{2}})'' + \frac{1}{r}(e^{\frac{A}{2}})' = -2K_0(e^{\frac{A}{2}})^3 + \mathcal{O}(r^n). \tag{44}$$

In order to have regular solutions at the origin, therefore A' must vanish.

We have obtained ten boundary conditions. Let us list them:

$$\begin{aligned} &R(0) = 0, \\ &P(0) = n, \\ &A(0) = 0 \qquad && A'(0) = 0, \\ &K(0) = 0 \qquad && K'(0) = 0, \\ &H(0) = 0 \qquad && H'(0) = 1, \end{aligned} \tag{45}$$

and

$$R(\infty) = 1 \qquad Q(\infty) = 0. \tag{46}$$

Numerical Solution

To solve the equations we use shooting method. We expand the functions around the origin as a power series of r. This requires introducing unknown constants. Then using the boundary conditions at the origin we try to match the two boundary conditions at infinity. This is achieved by adjusting these parameters.

The expansions of the functions are as follows

$$R(r) = R_0 r^n + \frac{R_0}{4(n+1)}[\beta^2 - Q_0^2 - nK_0(2Q_0 + nK_0)]r^{n+2}, \tag{47a}$$

$$P(r) = n - \frac{\beta n R_0^2}{n+1}(Q_0 + nK_0)r^{2(n+1)}, \tag{47b}$$

$$Q(r) = Q_0 - \beta R_0^2 r^{2n}, \tag{47c}$$

$$A(r) = -K_0^2 r^2 - \frac{1}{2}K_0^4 r^4 - \frac{\alpha R_0^2}{(n+1)^2}[\beta^2 - Q_0^2 + K_0^2(\frac{4n+1}{2n+1})]r^{2(n+1)}, \tag{47d}$$

$$K(r) = K_0 r^2 + K_0^3 r^4 + \frac{\alpha R_0^2}{(n+1)}[Q_0 - \frac{nK_0}{2n+1}]r^{2(n+1)}, \tag{47e}$$

$$H(r) = r - \frac{\alpha n R_0^2}{2n+1} r^{2n+1}, \tag{47f}$$

where R_0, Q_0 and K_0 are unknown parameters. We have one too many parameters since in exchange we have traded only two boundary conditions at infinity. To get rid of this problem we will examine Einstein's equations (16) at infinity and we will see that we can establish a relation between Q_0 and K_0.

The metric at infinity should approach to a metric generated by a point particle with mass m and spin J described by the line element[8]

$$ds^2 = (adt + \frac{4GJ}{a^3}d\varphi)^2 - B^2 r^2 d\varphi^2 - dr^2, \tag{48}$$

where $B = 1 - \frac{4Gm}{a^2}$ and a is an arbitrary constant. Let us remember the Eqs.(16)

$$R^a_{\ b} = \frac{1}{2\sqrt{-\lambda}}\frac{d}{dr}(\sqrt{-\lambda}\chi^a_{\ b}) = 8\pi G(T^a_{\ b} - T\delta^a_{\ b})$$

which, following integration, gives [7]

$$\sqrt{-\lambda}\chi^t_{\ t}(r) = -16\pi G\int_0^r \sqrt{-\lambda}(T^r_{\ r} + T^\varphi_{\ \varphi})dr \tag{49a}$$

$$\sqrt{-\lambda}\chi^t_{\ \varphi}(r) = 16\pi G\int_0^r \sqrt{-\lambda}T^t_{\ \varphi}dr \tag{49b}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ t}(r) = -2K_0 + 16\pi G\int_0^r \sqrt{-\lambda}T^\varphi_{\ t}dr \tag{49c}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ \varphi}(r) = 2 - 16\pi G\int_0^r \sqrt{-\lambda}(T^r_{\ r} + T^t_{\ t})dr \tag{49d}$$

where we used

$$\sqrt{-\lambda}\chi^t_{\ t}(0) = 0 \tag{50a}$$

$$\sqrt{-\lambda}\chi^t_{\ \varphi}(0) = 0 \tag{50b}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ t}(0) = -2K_0 \tag{50c}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ \varphi}(0) = 2 \tag{50d}$$

which we obtained by using eq.(25). We can similarly derive the behavior of $\chi^a_{\ b}$ at infinity using eq.(48),

$$\sqrt{-\lambda}\chi^t_{\ t} \to 0 \tag{51a}$$

$$\sqrt{-\lambda}\chi^t_{\ \varphi} \to -\frac{8GJB}{a^3} \tag{51b}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ t} \to 0 \tag{51c}$$

$$\sqrt{-\lambda}\chi^\varphi_{\ \varphi} \to 2aB. \tag{51d}$$

Thus Eqs.(49c) and (51c) yield the relation

$$K_0 = 8\pi G \int_0^\infty \sqrt{-\lambda} T^\varphi_t dr. \tag{52}$$

Using Eqs.(14) and (23) it is easy to see that

$$\begin{aligned} K_0 &= \frac{8\pi G\kappa}{e^2} \int_0^\infty Q \frac{dQ}{dr} dr \\ &= -\frac{\alpha}{2\beta} Q_0^2. \end{aligned} \tag{53}$$

This expression is useful in reducing the number of arbitrary parameters in the Eqs.(47a-f) from three to two.

Similarly by comparing Eqs.(49b)–(51b) we get an expression for the angular momentum,

$$J = -\frac{2\pi a^3}{B} \int_0^\infty \sqrt{-\lambda} T^t_{\ \varphi} dr. \tag{54}$$

Using Eqs.(13) and (22) we obtain

$$\begin{aligned} J &= \frac{2\pi a^3 \kappa}{Be^2} \int_0^\infty P \frac{dP}{dr} dr \\ &= -\frac{\pi a^3 \kappa n^2}{Be^2}. \end{aligned} \tag{55}$$

From the remaining equations in (49) and (51) we also get the following relations,

$$\int_0^\infty \sqrt{-\lambda}(T^r_{\ r} + T^\varphi_{\ \varphi}) dr = 0 \tag{56}$$

$$aB = 1 - 8\pi G \int_0^\infty \sqrt{-\lambda}(T^t_{\ t} + T^r_{\ r}) dr. \tag{57}$$

We can find, by using the Eqs.(13),(14),(22),(23) and (56), that

$$T^t_t + T^r_r = -\frac{v^2}{\beta}\frac{1}{\sqrt{-\lambda}}\frac{d}{dr}(PQ). \tag{58}$$

Putting this into the eq.(57) gives

$$aB = 1 - \frac{\alpha n Q_0}{\beta}. \tag{59}$$

From eq(48) we see that

$$K(\infty) = \frac{4GJ}{a^4}. \tag{60}$$

Using eqs(55) and (59) we get

$$K(\infty) = -\frac{\alpha n^2}{2(\beta - \alpha n Q_0)}. \tag{61}$$

We will not use (59) and (61) in our numerical calculations. They are nonetheless useful to check the accuracy of our numerical analysis. We find that numerical results agree with these equations.

We give the results of our numerical analysis in the Figures 1-8. All the solutions are obtained for $n = 1$ and $\beta = 0.1$. Figures 1-3 show the results for the scalar and the gauge fields. In the Figure 4 we plot the metric function $e^A \equiv g_{tt}$. For values $\alpha < 0.75$ e^A approaches to a non-zero constant at spatial infinity. For $\alpha > 0.75$ its value approaches 0 (for $\alpha = 0.8$, for example, e^A approaches zero around $r \approx 125$). Figure 5 shows the results for the metric function $K(r)$. Plots of $g_{\varphi\varphi}$ is given in Figure 6. From Figure 7 we see that $e^{\frac{D}{2}} \to Br$ except for values $\alpha > 0.75$ for which it goes to a constant. To see this better we plot $e^{\frac{D}{2}}$ around $\alpha = 0.75$. This result is different than supermassive solutions of Abelian Higgs model [9].

We obtained the equations (59) and (61) by assuming that the asymptotic form of metric is Eq. (48). It is clear that supermassive solitons do not have these asymptotics. Therefore Eq. (59) is not valid for $\alpha > 0.75$. But we find that Eq. (61) holds true for all values of α. We found no discrepancy between numerical results and Eq. (61).

Closed Time-like Curves

It is a well-known fact that some matter distributions allow closed time-like curves. First such space-time solution which has a constant and uniform energy density was obtained by Gödel [10]. It is generally believed that these matter distributions are unphysical. This is known as "Chronology Protection Conjecture." Recently it has been shown that in the case of rotationally symmetric open universe weak energy combined with the absence of closed time-like curves at spatial infinity prevents the existence of closed time-like curves everywhere [11].

The metric (48) also supports closed time-like curves for sufficiently small radius. For

$$r < \frac{4GJ}{a^3 B} \tag{62}$$

φ becomes a cyclic time-like coordinate, hence leading to closed time-like curves. Our numerical results for the space-time of the Chern-Simons solitons show that φ does not become time-like, i.e., $g_{\varphi\varphi}$ does not become positive, for the cases we have considered, as can be seen in Figure 6. Therefore these spinning soliton solutions do not create a space-time which supports closed time-like curves.

Conclusions

We have solved for the Chern-Simons Solitons in $2+1$ dimensional Einstein gravity. We have obtained numerical solutions for the space-time created by these solitons. The results show that this space-time does not allow the existence of closed time-like curves. This is a result expected for physical matter distributions and it is known as the Chronology Protection Conjecture. We have also found solutions for the supermassive solitons which make the space-time closed.

References

[1] J. Hong, Y. Kim and P. Y. Pac, Phys. Rev. Lett. **64** (1990)2230

[2] R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. **64** (1990)2234

[3] R. Jackiw, K. Lee and E. J. Weinberg, Phys. Rev. **D42** (1990)3488

[4] G. Clément, Ann. Phys. (N.Y.) **201** (1990)241

[5] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, 1972

[6] L. D. Landau and Lifshitz, *The Classical Theory of Fields*, 4th Revised English Edition, Pergamon Press, 1975

[7] B. Linet, Gen. Relativ. Gravit. **22** (1990)469

[8] S. Deser and R. Jackiw, Comm. Nucl. Part. Phys. **20**(1992)337

[9] M. Ortiz, Phys. Rev. **D43** (1991)2521

[10] K. Gödel, Rev. Mod. Phys. **21** (1949)447

[11] P. Menotti and D. Seminara, Preprints IFUP-TH-18/93 and IFUP-TH-43/93

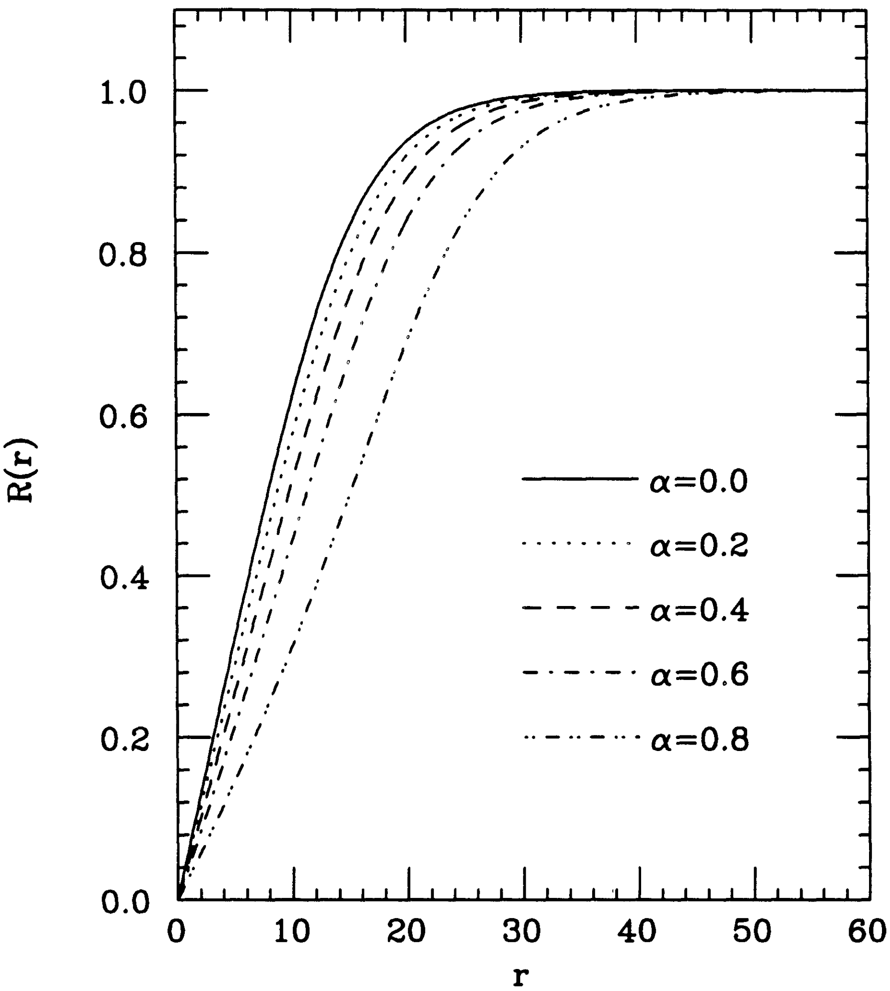

Figure 1. The plot of the solutions of the scalar field R(r) for $n = 1$, $\beta = 0.1$ and $\alpha = 0.0, 0.2, 0.4, 0.6, 0.8$.

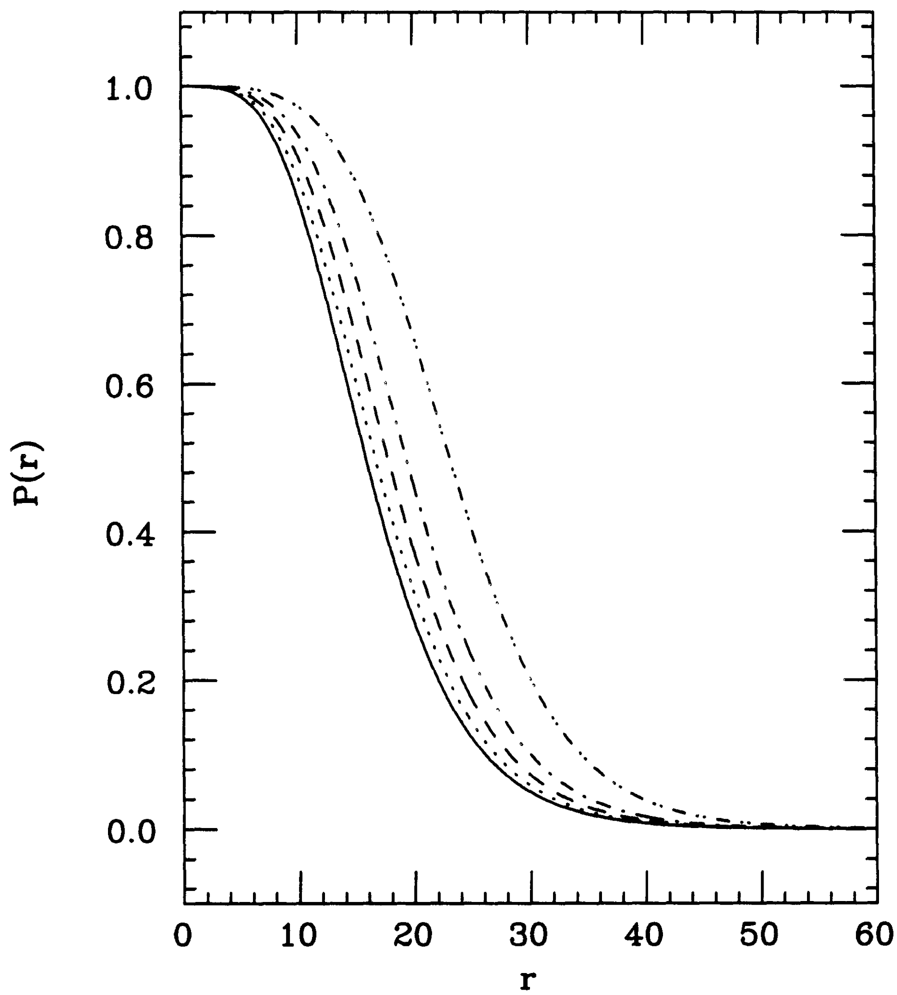

Figure 2. The plot of the solutions of the gauge field P(r). The conventions are the same as in Figure 1.

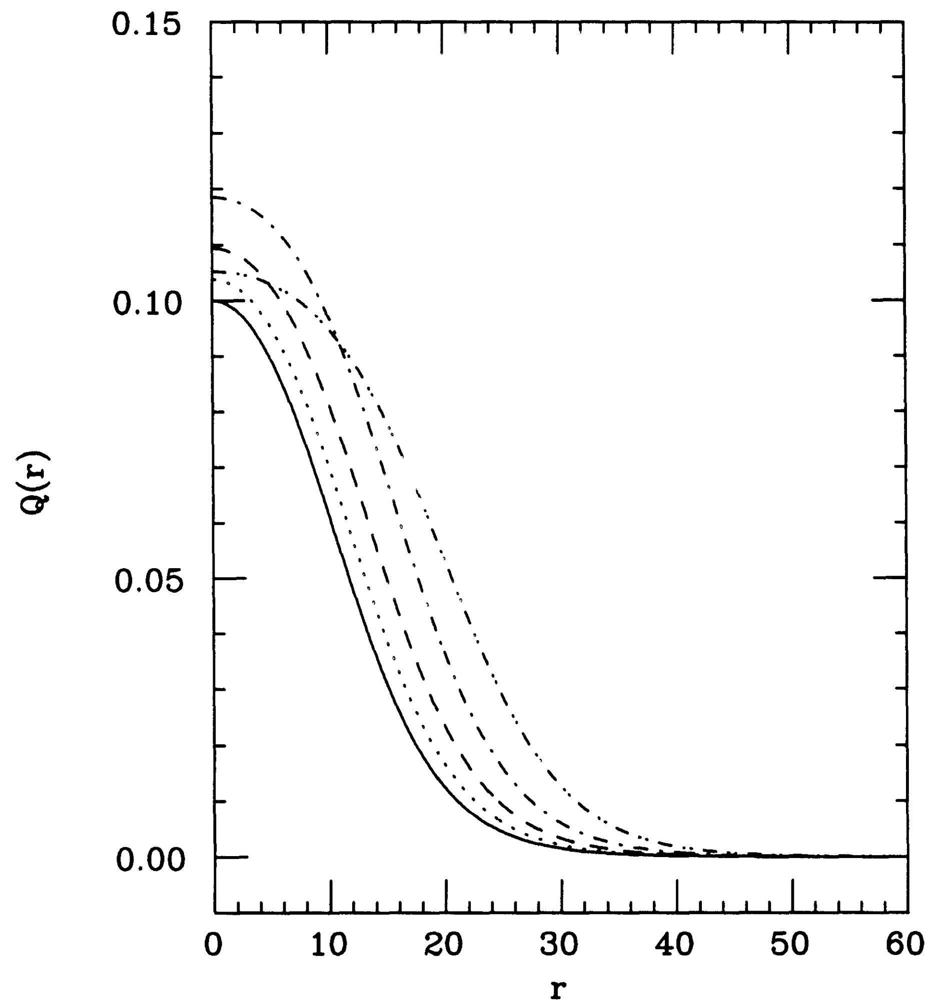

Figure 3. The plot of the solutions of the gauge field Q(r). The conventions are the same as in Figure 1.

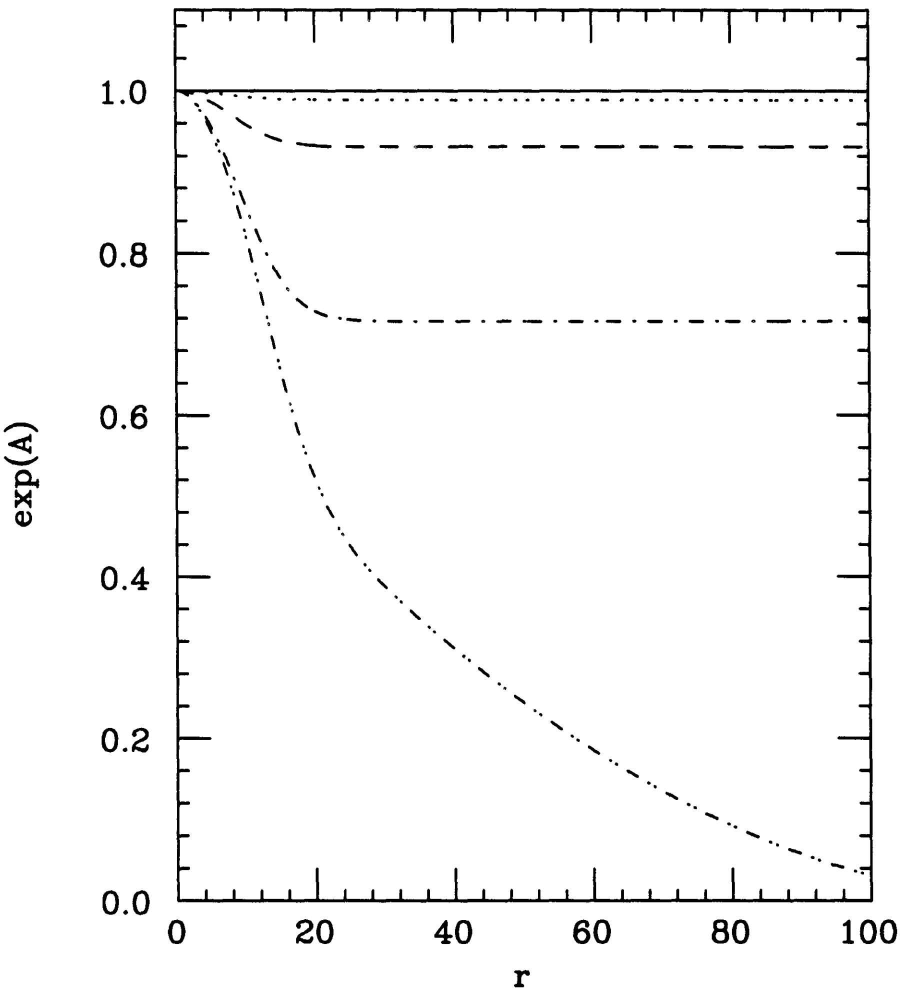

Figure 4. The plot of the solutions of the metric field e^A. The conventions are the same as in Figure 1.

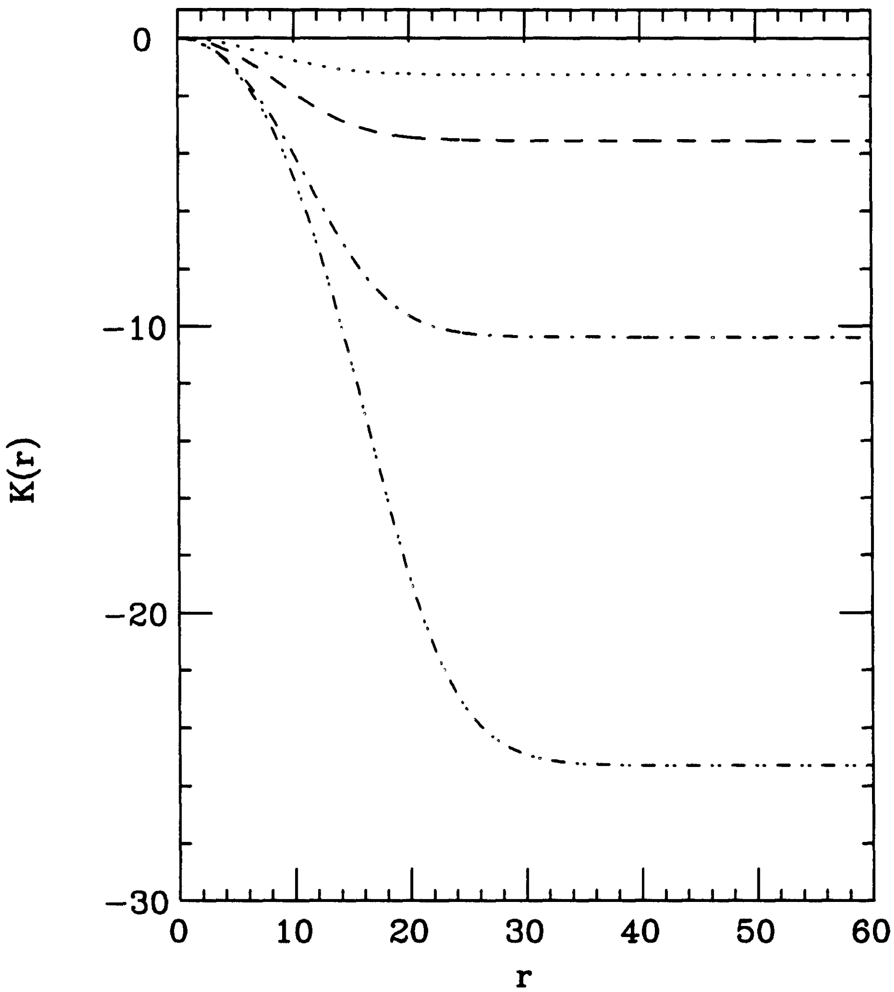

Figure 5. The plot of the solutions of the metric field K(r). The conventions are the same as in Figure 1.

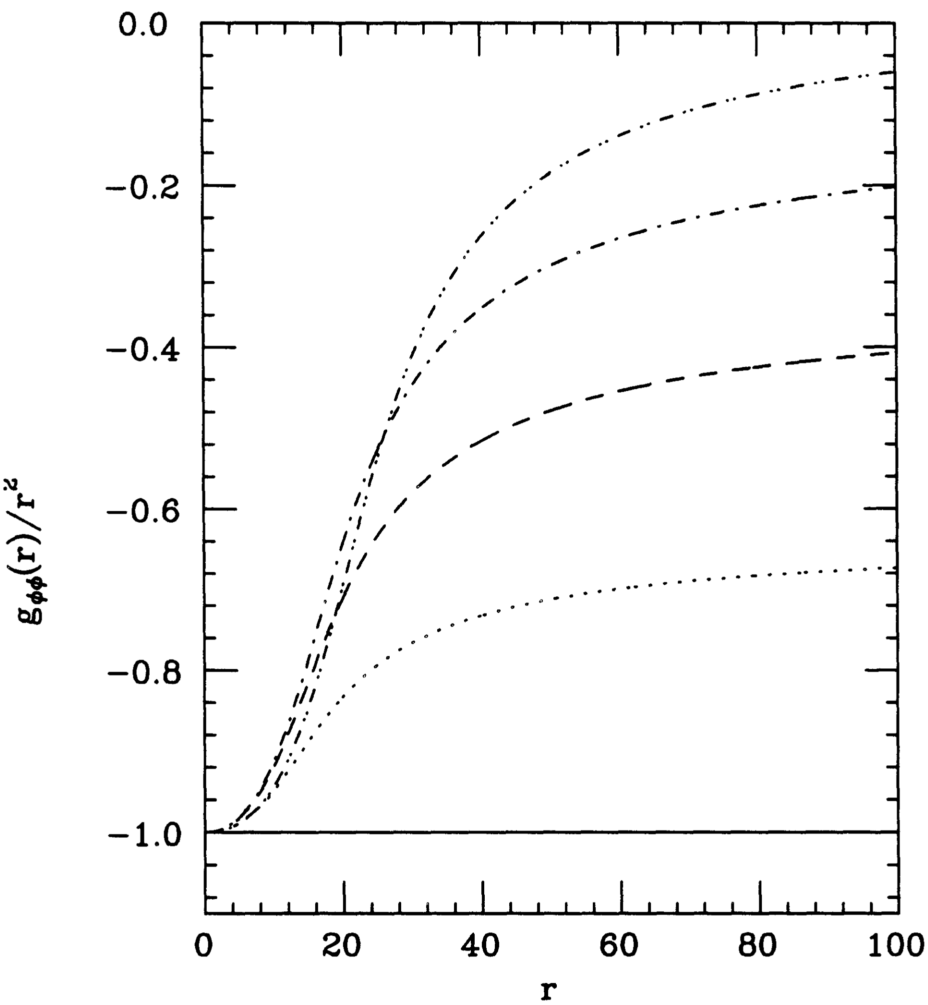

Figure 6. The plot of the solutions of the metric component $g_{\varphi\varphi}$ divided by the radius squared. This figure shows that $g_{\varphi\varphi}$ is negative everywhere, i.e., φ is a space-like coordinate everywhere. Hence there exists no closed time-like curves. The conventions are the same as in Figure 1.

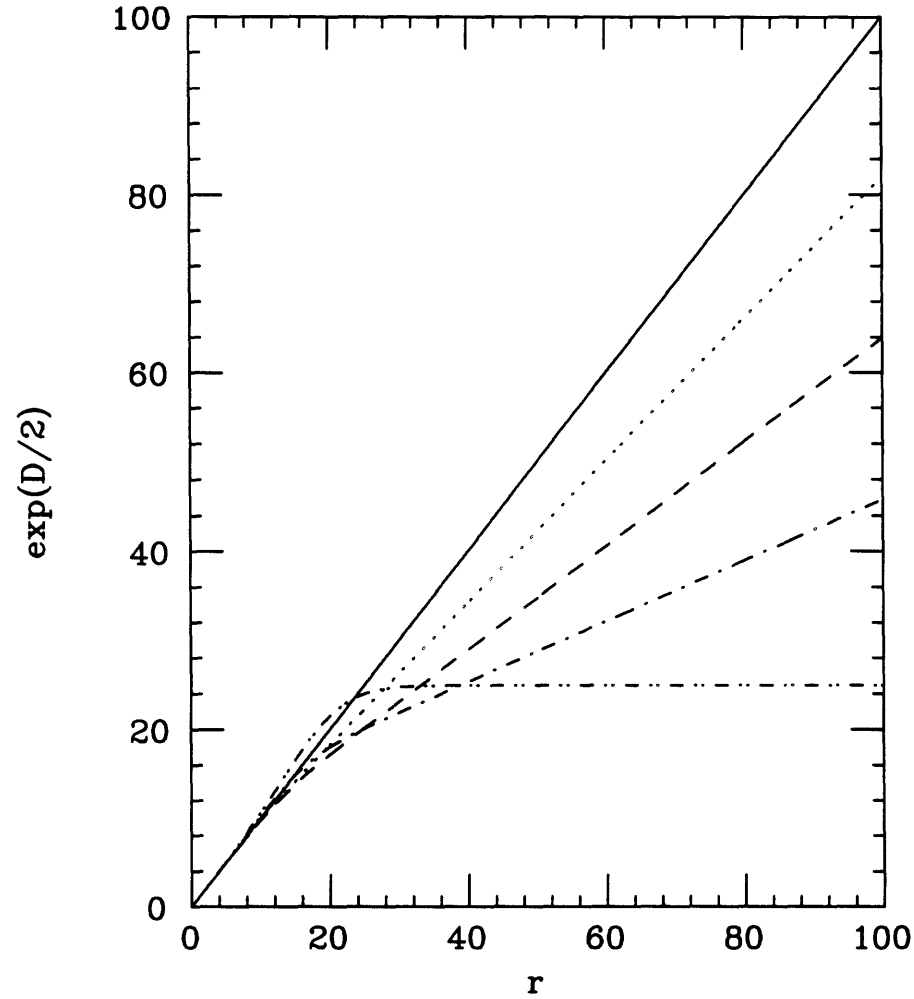

Figure 7. The plot of the solutions of the metric field $e^{D/2}$. The conventions are the same as in Figure 1.

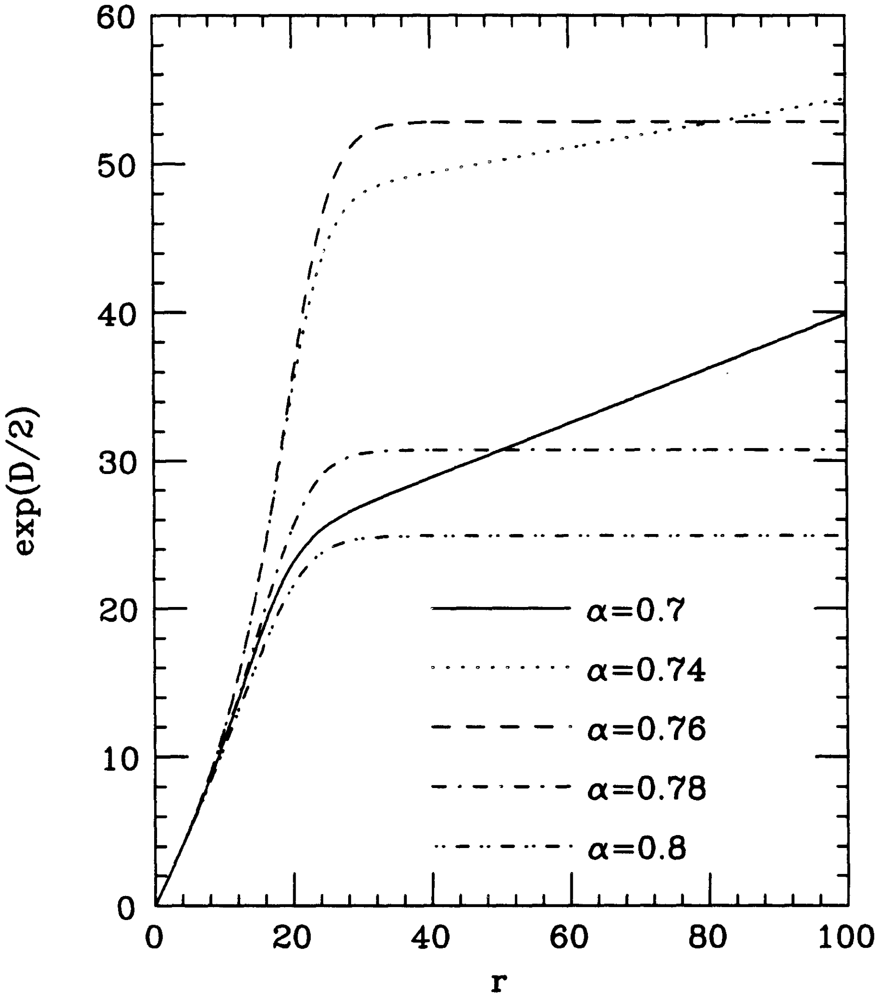

Figure 8. The plot of the solutions of the metric field $e^{D/2}$ for values of α around 0.75.